ON THE COHOMOLOGY OF MOMENT-ANGLE COMPLEXES ASSOCIATED TO GORENSTEIN* COMPLEXES

FEIFEI FAN AND XIANGJUN WANG

ABSTRACT. The main goal of this article is to study the cohomology rings and their applications of moment-angle complexes associated to Gorenstein* complexes, especially, the applications in combinatorial commutative algebra and combinatorics. First, we give a topological characterization of Gorenstein* complexes in terms of Alexander duality (as an application we give a topological proof of Stanley's Theorem). Next we give some cohomological transformation formulae of $\mathcal{Z}_K$, which are induced by some combinatorial operations on the Gorenstein* complex K, such as the connected sum operation and stellar subdivisions. We also prove that $\mathcal{Z}_K$ is a prime manifold whenever K is a flag 2-sphere by proving the indecomposability of their cohomology rings. Then we use these results to give the unique decomposition of the cohomology rings of moment-angle manifolds associated to simplicial 2-spheres, and explain how to use it to detect the cohomological rigidity problem of these moment-angle manifolds.

1. INTRODUCTION AND MAIN RESULTS

In 1990's Davis and Januszkiewicz [16] introduced quasi-toric manifolds over a simple polytope $\mathcal{P}$, a topological generalization of projective toric varieties which were being studied intensively by algebraic geometers. They observed that every quasi-toric manifold is the quotient of the manifold $\mathcal{Z}_{\mathcal{P}}$ constructed from the same polytope $\mathcal{P}$ (now called *moment-angle manifold*) by the free action of a real torus. Buchstaber and Panov [10] generalized this construction to any simplicial complex K, and named it the *moment-angle complex associated to K*. It has been actively studied in toric topology and has many connections with symplectic and algebraic geometry, and combinatorics.

Throughout this paper, we use the common notation **k** for the ground ring, which is always assumed to be the ring $\mathbb{Z}$ of integers or a field, and whenever there is no confusion, we use the tensor product notation $\otimes$ for $\otimes_{\mathbf{k}}$. Given a simplicial complex K, there is an associated ring known as the Stanley-Reisner ring of K, denoted $\mathbf{k}(K)$. The ring $\mathbf{k}(K)$ is a quotient of a finitely generated polynomial ring $\mathbf{k}[v_1, \ldots, v_m]$ with generators v_i for each vertex of K. Hochster [27], in purely algebraic work, calculated the Tor-modules

$\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$ in terms of the full subcomplexes of K. Buchstaber-Panov [10] proved that the cohomology algebra of $\mathcal{Z}_K$ is isomorphic to $\mathrm{Tor}_{\mathbf{k}[v_1,\dots,v_m]}(\mathbf{k}(K),\mathbf{k})$. Using the cellular cochain algebra model for $\mathcal{Z}_K$, Baskakov [5] gave an explicit formula for the cup product in $H^*(\mathcal{Z}_K,\mathbf{k})$, in terms of pairings between full subcomplexes (However, there is no proof in this very short paper. A complete proof of this is given by Bosio-Meersseman [7]). Other works on the cohomology ring of $\mathcal{Z}_K$ can be found in [2, 3, 7, 17, 21, 22, 36, 39].

This paper is a study of the ring structure of $H^*(\mathcal{Z}_K)$ and its applications to detecting the topology of $\mathcal{Z}_K$. More precisely, the paper is organized as follows.

§2 is a review of notations and previous results on moment-angle complexes. In §3, we introduce the Gorenstein* complexes and some of their algebraic and topological properties. We prove that (Theorem 3.4): a simplicial complex K is a generalized homology sphere if and only if it satisfies Alexander duality between full subcomplexes. Then by using the Poincaré duality of Gorenstein rings and the theory of moment-angle complexes, we give a topological proof for Stanley's Theorem (Theorem 3.3). In §4, we give a formula to calculate the cohomology ring of the moment angle complex associated to the connected sum of two Gorenstein* complexes. The main result of this section is:

Theorem 1.1 (Theorem 4.2)**.** *Let K_1 and K_2 be two $(n-1)$-dimensional $(n \geq 2)$ Grenstein* complex over $\mathbf{k}$ with m_1 and m_2 vertices respectively, and let $K = K_1 \# K_2$. Then the reduced cohomology ring of $\mathcal{Z}_K$ is given by the isomorphism*

$$\widetilde{H}^*(\mathcal{Z}_K;\mathbf{k}) \cong \mathcal{R}(K_1,K_2;\mathbf{k})/\mathcal{I}(K_1,K_2;\mathbf{k}),$$

where

$$\mathcal{R}(K_1,K_2;\mathbf{k}) = G^{m_2-n}(\widetilde{H}^*(\mathcal{Z}_{K_1};\mathbf{k})) \times G^{m_1-n}(\widetilde{H}^*(\mathcal{Z}_{K_2};\mathbf{k})) \times \widetilde{H}^*(M;\mathbf{k}),$$

(see Construction 4.1 for the operation G)

$$M = \overset{m_1+m_2-2n}{\underset{i=2}{\sharp}} \lambda(i)(S^{i+1} \times S^{m_1+m_2-i-1}),$$

$$\lambda(i) = \tbinom{m_1+m_2-2n}{i} - \tbinom{m_1-n}{i} - \tbinom{m_2-n}{i},$$

$\mathcal{I}(K_1,K_2;\mathbf{k})$ is an ideal of $\mathcal{R}(K_1,K_2;\mathbf{k})$ generated by

$$([Z_1],0,0) - (0,0,[M]) \ \ and \ \ (0,[Z_2],0) - (0,0,[M]),$$

where $[Z_1]$ (resp. $[Z_2]$, $[M]$) is the top dimensional generator of $G^{m_2-n}(\widetilde{H}^*(\mathcal{Z}_{K_1};\mathbf{k}))$ (resp. $G^{m_1-n}(\widetilde{H}^*(\mathcal{Z}_{K_2};\mathbf{k}))$, $\widetilde{H}^*(M;\mathbf{k})$).

Theorem 1.1 leads to a conjecture on the topology of $\mathcal{Z}_{K_1\#K_2}$:

Conjecture 1.2. If K_1, K_2 are generalized homology spheres, then $\mathcal{Z}_{K_1\#K_2}$ is homeomorphic to

$$\mathcal{G}^{m_2-n}(\mathcal{Z}_{K_1}) \# \mathcal{G}^{m_1-n}(\mathcal{Z}_{K_2}) \# \overset{m_1+m_2-2n}{\underset{i=2}{\sharp}} \lambda(i)(S^{i+1} \times S^{m_1+m_2-i-1}),$$

where $\mathcal{G}$ is an operation on manifolds (See Definition 4.6) first defined and studied by González Acuña.

In §5, we study the cohomology ring of the moment angle complex associated to the stellar subdivision $\mathrm{S}_\sigma K$ of a given simplicial sphere K. Although in general it is difficult to describe the relation between $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ and $H^*(\mathcal{Z}_K)$, there is a simple formula to calculate $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ when the simplex σ satisfies some local conditions. That is:

Theorem 1.3 (Theorem 5.4). *Let K be a Corenstein* complex of dimension $n-1$ with m vertices, $\sigma \in K$ be a simplex of dimension q. Let $\mathcal{V}$ be the vertex set of $\mathrm{link}_K \sigma$, $s = |\mathcal{V}| + |\sigma|$. If for any $I \subset \mathcal{V}$, the inclusion map $\varphi_I : (\mathrm{link}_K \sigma)_I \to K_I$ is nullhomotopic, and if one of the following additional conditions are satisfied:*

(a) $\mathbf{k}$ *is a field.*
(b) $\mathbf{k} = \mathbb{Z}$ *and $H^*(\mathcal{Z}_{\mathrm{link}_K \sigma})$ is torsion free.*

Then the cohomology ring of $\mathcal{Z}_{\mathrm{S}_\sigma K}$ is given by the isomorphism

$$\widetilde{H}^*(\mathcal{Z}_{\mathrm{S}_\sigma K}; \mathbf{k}) \cong \widetilde{H}^*(\mathcal{G}(\mathcal{Z}_K) \# Y; \mathbf{k}),$$

where

$$Y = \underset{i+j \geq 1}{\sharp} f_i \cdot \binom{m-s}{j}(S^{i+j+2} \times S^{m+n-i-j-1}),$$

$$f_i = \mathrm{rank}\, H^i(\mathcal{Z}_{\mathrm{link}_K \sigma}).$$

In §6, we study the moment-angle manifolds associated to flag 2-spheres, and get an important algebraic property of $H^*(\mathcal{Z}_K)$ when K is a flag 2-sphere, which implies a topological property of $\mathcal{Z}_K$. They are shown by the following

Theorem 1.4 (Theorem 6.6). *Let K be a flag 2-sphere. Then $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is a graded indecomposable ring, where $[\mathcal{Z}_K]$ is the top class of $\widetilde{H}^*(\mathcal{Z}_K)$.*

Theorem 1.5 (Theorem 6.8). *Let K be a flag 2-sphere. Then $\mathcal{Z}_K$ is a prime manifold.*

In §7, we further discuss the cohomology ring $H^*(\mathcal{Z}_K)$ when K is a simplicial 2-sphere. By using the results in §4 and §6, we get the following

Theorem 1.6 (Theorem 7.4). *Let K and K' be simplicial 2-spheres. If*

$$K = K_1 \# K_2 \#, \ldots, \# K_n, \quad K' = K'_1 \# K'_2 \#, \ldots, \# K'_{n'}$$

such that each K_i and K'_i are irreducible, and if $H^*(\mathcal{Z}_K; \mathbf{k}) \cong H^*(\mathcal{Z}_{K'}; \mathbf{k})$ (as graded rings), then $n = n'$ and there is a permutation $j \curvearrowright j'$ such that

$$H^*(\mathcal{Z}_{K_j}; \mathbf{k})/([\mathcal{Z}_{K_j}]) \cong H^*(\mathcal{Z}_{K'_{j'}}; \mathbf{k})/([\mathcal{Z}_{K'_{j'}}]), \; 1 \leq j \leq n.$$

This theorem together with Conjecture 1.2 provides a heuristic method for solving the cohomological rigidity problem of moment-angle manifolds associated to simplicial 2-spheres.

2. Preliminaries

2.1. simplicial complexes and face rings. Let $\mathcal{S}$ be a finite set. Given a subset $\sigma \subseteq \mathcal{S}$, we denote its cardinality by $|\sigma|$.

Definition 2.1. An *abstract simplicial complex* on a set $\mathcal{S}$ is a collection $K = \{\sigma\}$ of subsets of $\mathcal{S}$ such that for each $\sigma \in K$ all subsets of σ (including $\varnothing$) also belong to K. An subset $\sigma \in K$ is called a *simplex* of K. A maximal simplex is also called a *facet*.

One-element simplices are called *vertices* of K. If K contains all one-element subsets of $\mathcal{S}$, then we say that K is a simplicial complex on the vertex set $\mathcal{S}$.

It is sometimes convenient to consider simplicial complexes K whose vertex sets are proper subsets of $\mathcal{S}$. In this case we refer to a one-element subset of $\mathcal{S}$ which is not a vertex of K as a *ghost vertex*.

The *dimension* of a simplex $\sigma \in K$ is $\dim\sigma = |\sigma| - 1$. The dimension of an abstract simplicial complex is the maximal dimension of its simplices. A simplicial complex K is *pure* if all its facets have the same dimension.

In most construction of this paper it is safe to fix an ordering in $\mathcal{S}$ and identify $\mathcal{S}$ with the index set $[m] = \{1, \ldots, m\}$. We denote by Δ^m the abstract simplicial complex $2^{[m+1]}$ consisting of all subsets of $[m+1]$, and denote by $\partial\Delta^m$ the simplicial complex $2^{[m+1]} \setminus [m+1]$ (the boundary of Δ^m).

Definition 2.2. Let K_1 and K_2 be simplicial complexes on sets $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. The *join* of K_1 and K_2 is the simplicial complex

$$K_1 * K_2 = \{\sigma \subseteq \mathcal{S}_1 \sqcup \mathcal{S}_2 : \sigma = \sigma_1 \cup \sigma_2,\ \sigma_1 \in K_1, \sigma_2 \in K_2\}.$$

Definition 2.3. Let K_1, K_2 be two pure n-dimensional simplicial complexes on sets $\mathcal{S}_1$, $\mathcal{S}_2$ respectively. Suppose we are given two facets $\sigma_1 \in K_1$, $\sigma_2 \in K_2$. Fix an identification of σ_1 and σ_2 (by identifying their vertices), and denote by $\mathcal{S}_1 \cup_\sigma \mathcal{S}_2$ the union of $\mathcal{S}_1$ and $\mathcal{S}_2$ with σ_1 and σ_2 identified. Then the simplicial complex $(K_1 \cup K_2) \setminus \sigma$ is called a *connected sum* of K_1 and K_2. It depends on the way of choosing the two simplices and identifying their vertices. Let $\mathcal{C}(K_1 \# K_2)$ denote the set of connected sums of K_1 and K_2.

Definition 2.4. Let K be an abstract simplicial complex on $[m]$.

(1) Given a subset $I \subseteq [m]$, define $K_I \subseteq K$ to be the *full sub-complex* of K consisting of all simplices of K which have all of their vertices in I, that is

$$K_I = \{\sigma \cap I \mid \sigma \in K\}.$$

(2) For a simplex σ of K, the *link* and the *star* of σ are the simplicial subcomplexes

$$\mathrm{link}_K \sigma = \{\tau \in K : \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\};$$
$$\mathrm{star}_K \sigma = \{\tau \in K : \sigma \cup \tau \in K\}.$$

(3) The simplicial complex $\Delta^0 * K$ (the join of K and a point) is called the *cone* over K.

Definition 2.5. A simplicial complex K is called a *flag complex* if each of its missing faces has two vertices.

Definition 2.6. Let K be a simplicial complex with vertex set $[m]$. A *missing face* of K is a sequence $(i_1, \ldots, i_k) \subseteq [m]$ such that $(i_1, \ldots, i_k) \notin K$, but every proper subsequence of $(i_1, \ldots, i_k)$ is a simplex of K. Denote by $MF(K)$ the set of all missing faces of K.

We use the common notation $\mathbf{k}$ to denote a field or the ring $\mathbb{Z}$ of integers. Let $\mathbf{k}[m] = \mathbf{k}[v_1, \ldots, v_2]$ denote the graded polynomial algebra on m variables, $\deg(v_i) = 2$.

Definition 2.7. The *face ring* (also known as the *Stanley-Reisner ring*) of a simplicial complex K on a set $[m]$ is the quotient ring

$$\mathbf{k}(K) = \mathbf{k}[m]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_s}$ for which $(i_1, \ldots, i_s)$ is not a simplex in K. We refer to $\mathcal{I}_K$ as the Stanley-Reisner ideal of K.

2.2. **moment-angle complexes.** We start with a generalization of the notion of moment-angle complex.

Definition 2.8. Let $(\underline{X}, \underline{A})$ denote a collection of based CW pairs $\{(X_i, A_i, x_i)\}_{i=1}^m$. Given a simplicial complex K on $[m]$ (may have ghost vertices), the *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is defined to be:

$$\mathcal{Z}_K^{[m]}(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} D(\sigma),$$

where

$$D(\sigma) = \prod_{i=1}^m Y_i \quad \text{and} \quad Y_i = \begin{cases} X_i & \text{if} \quad i \in \sigma, \\ A_i & \text{if} \quad i \notin \sigma. \end{cases}$$

If $[m]$ is just the vertex set of K, the notation $\mathcal{Z}_K^{[m]}(\underline{X}, \underline{A})$ is simplified to $\mathcal{Z}_K(\underline{X}, \underline{A})$).

In the case when $(\underline{X}, \underline{A}) = (D^2, S^1)$, we obtain the usual *moment-angle complex*, $\mathcal{Z}_K^{[m]} = \mathcal{Z}_K^{[m]}(D^2, S^1)$, and in the case $(\underline{X}, \underline{A}) = (D^1, S^0)$, $\mathcal{R}\mathcal{Z}_K^{[m]} = \mathcal{Z}_K^{[m]}(D^1, S^0)$ is called the *real moment-angle complex*.

Now suppose that K is an $(n-1)$-dimentional simplicial sphere (a triangulation of a sphere) with m vertices. Then, as shown by Buchstaber and Panov [10], the moment-angle complex $\mathcal{Z}_K$ is a manifold of dimension $n + m$. Furthermore, Cai [13] gave a necessary and sufficient condition for $\mathcal{Z}_K$ to be a topological manifold.

Definition 2.9. A space X is a *homology n-manifold* if it has the same local homology groups as $\mathbb{R}^n$, i.e., if for each $x \in X$,

$$H_i(X, X - x) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

X is *orientable* if there is a function $x \mapsto \mu_x$ assigning to each $x \in X$ a choice of generator μ_x of $H_n(X, X - x)$ satisfies that for each $x \in X$ there is an open neighborhood U and a class $\alpha \in H_n(X, X - U)$ such that the map induced by the inclusion $(X, X - U) \to (X, X - x)$ maps α to μ_x for each $x \in U$.

A space X is a *generalized homology n-sphere* (for short, a "GHS") if it is a homology n-manifold with the same homology as S^n ('generalized' because a homology sphere is usually assumed to be a manifold).

Definitions 2.9 can be extended in an obvious fashion to define a homology manifold or GHS over $\mathbf{k}$. For a simplicial complex K,

Theorem 2.10. [13, Corollary 2.10] *Let K be a simplicial complex with m vertices. Then the moment-angle complex $\mathcal{Z}_K$ is a topological $(n + m)$-manifold if and only if $|K|$ is a generalized homology $(n - 1)$-sphere.*

In the above case, $\mathcal{Z}_K$ is referred to as a *moment-angle manifold*. In particular, if K is a polytopal sphere (a triangulated sphere isomorphic to the boundary complex of a simplicial polytope), or more generally a starshaped sphere (see Definition 2.11), then $\mathcal{Z}_K$ admits a smooth structure [37].

Definition 2.11. A simplicial sphere K of dimension n is said to be *starshaped* if there is a geometric realization $|K|$ of K in $\mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ with the property that each ray emanating from p meets $|K|$ in exactly one point.

2.3. **Cohomology rings of moment-angle complexes.** In this paper we mainly use the following results to calculate the cohomology ring of $\mathcal{Z}_K$, which is proved by Buchstaber and Panov [11, Theorems 7.6] for the case over a field, [6] for the general case. Another proof of Theorem 2.12 for the case over $\mathbb{Z}$ was given by Franz [22].

Theorem 2.12 (Buchstaber-Panov, [36, Theorem 4.7])**.** *Let K be a abstract simplicial complex on the set $[m]$. Then the cohomology algebra of the moment-angle complex $\mathcal{Z}_K^{[m]}$ is given by the isomorphisms*

$$H^*(\mathcal{Z}_K^{[m]}; \mathbf{k}) \cong \operatorname{Tor}^{*,*}_{\mathbf{k}[m]}(\mathbf{k}(K), \mathbf{k}) \cong \bigoplus_{I \subseteq [m]} \widetilde{H}^*(K_I; \mathbf{k})$$

where

$$H^p(\mathcal{Z}_K^{[m]}; \mathbf{k}) \cong \bigoplus_{\substack{J \in [m] \\ -i + 2|J| = p}} \operatorname{Tor}^{-i, 2J}_{\mathbf{k}[m]}(\mathbf{k}(K), \mathbf{k})$$

and

$$\operatorname{Tor}^{-i, 2J}_{\mathbf{k}[m]}(\mathbf{k}(K), \mathbf{k}) \cong \widetilde{H}^{|J|-i-1}(K_J; \mathbf{k}).$$

We assume $\widetilde{H}^{-1}(\varnothing; \mathbf{k}) = \mathbf{k}$ above.

Definition 2.13. Since $\mathrm{Tor}^{*,*}_{\mathbf{k}[m]}(\mathbf{k}(K), \mathbf{k})$ is a graded algebra, the isomorphisms of Theorem 2.12 turn the direct sum $\bigoplus_{I \subseteq [m]} \widetilde{H}^*(K_I; \mathbf{k})$ into a graded $\mathbf{k}$-algebra, called the *Hochster algebra* of K on [m] and denoted $\mathcal{H}^*_{[m]}(K; \mathbf{k})$ (if [m] is the vertex set of K, simplify it to $\mathcal{H}^*(K; \mathbf{k})$), where $\mathcal{H}^i_{[m]}(K; \mathbf{k}) = \bigoplus_{J \subseteq [m]} \widetilde{H}^{i-1}(K_J; \mathbf{k})$. Let

$$\widetilde{\mathcal{H}}^*_{[m]}(K) = \bigoplus_{J \subseteq [m],\, J \neq \varnothing} \widetilde{H}^*(K_J),$$

called the *reduced Hochster algebra* of K.

Given two elements $u, v \in \mathcal{H}^*_{[m]}(K)$, denote by $u * v$ the product of u and v in $\mathcal{H}^*_{[m]}(K)$, called the *star product*.

So $\mathcal{H}^*_{[m]}(K; \mathbf{k})$ is an augmented $\mathbf{k}$-algebra with $\widetilde{\mathcal{H}}^*_{[m]}(K; \mathbf{k})$ as its augmentation ideal. Theorem 2.12 gives that

$$H^*(\mathcal{Z}^{[m]}_K; \mathbf{k}) \cong \mathcal{H}^*_{[m]}(K; \mathbf{k}),$$

and

$$\widetilde{H}^*(\mathcal{Z}^{[m]}_K; \mathbf{k}) \cong \widetilde{\mathcal{H}}^*_{[m]}(K; \mathbf{k}).$$

Let $\widetilde{C}^q(K; \mathbf{k})$ denote the qth reduced simplicial cochain group of K with coefficients in $\mathbf{k}$. For a oriented (ordered) simplex $\sigma = (i_1, \ldots, i_p)$ of K, denote still by $\sigma \in \widetilde{C}^{p-1}(K; \mathbf{k})$ the basis cochain corresponding to σ; it takes value 1 on σ and vanishes on all other simplices.

Definition 2.14. Define a $\mathbf{k}$-bilinear *union product* in the simplicial cochains of full subcomplexes of K

$$\sqcup : \widetilde{C}^{p-1}(K_I; \mathbf{k}) \otimes \widetilde{C}^{q-1}(K_J; \mathbf{k}) \to \widetilde{C}^{p+q-1}(K_{I \cup J}; \mathbf{k}), \quad p, q \geq 0$$

$$\sigma \otimes \tau \mapsto \sigma \sqcup \tau$$

by setting $\sigma \sqcup \tau$ to be the juxtaposition of σ and τ if $I \cap J = \varnothing$ and $\sigma \cup \tau$ is a simplex of $K_{I \cup J}$; zero otherwise.

Similarly, the *excision product* in the simplicial chains and cochains of full subcomplexes is defined by

$$\sqcap : \widetilde{C}_{p+q-1}(K_I; \mathbf{k}) \otimes \widetilde{C}^{p-1}(K_J; \mathbf{k}) \to \widetilde{C}_{q-1}(K_{I \setminus J}; \mathbf{k}), \quad p, q \geq 0$$

$$\sigma \otimes \tau \mapsto \sigma \sqcap \tau$$

Here $\sigma \sqcap \tau = \varepsilon_{\sigma, \tau}(\sigma \setminus \tau)$ if $J \subset I$, $\tau \subset \sigma$ and $\sigma \setminus \tau \subset I \setminus J$; zero otherwise, and $\varepsilon_{\sigma, \tau}$ is the sign of the permutation sending $\tau \sqcup (\sigma \setminus \tau)$ to σ.

It is easily verified that the union product of cochains induces a union product of cohomology classes in the Hochster algebra of K:

$$\sqcup : \mathcal{H}^p(K; \mathbf{k}) \otimes \mathcal{H}^q(K; \mathbf{k}) \to \mathcal{H}^{p+q}(K; \mathbf{k}), \quad p, q \geq 0.$$

Similarly, there is an induced excision product in homology and cohomology of the full subcomplexes of K. Union and excision product are related by the formula

$$\psi(c \sqcap \phi) = (\phi \sqcup \psi)(c)$$

for $c \in \widetilde{C}_{p+q-1}(K_I)$, $\phi \in \widetilde{C}^{p-1}(K_J)$ and $\psi \in \widetilde{C}^{q-1}(K_{I \setminus J})$. Intuitively, the union product (resp. excision produc) is an analogue of cup product (resp. cap product) in cohomology (resp. homology and cohomology) of a space. Actually, the union and excision product for K do respectively correspond (up to a sign) to the cup and cap product for $\mathcal{Z}_k$. In §3 we will futher show that like the Poincaré duality of manifolds is given by the cap product map, the Alexander duality of Gorenstein* complexes is given by the excision product map.

Baskakov in [5] asserted that the star product and the union product are identical. Howerver there is a sign difference between these two products. This was pointed out by Bosio and Meersseman [7, Theorem 10.1] (Buchstaber and Panov also indicated this defect [12]).

Theorem 2.15 ([12, Proposition 3.2.10]). *Let K be a simplicial complex on $[m]$. Given cohomology classes $[u] \in \widetilde{H}^{p-1}(K_I)$ and $[v] \in \widetilde{H}^{q-1}(K_J)$ with $I, J \in [m]$, then*

$$[u] * [v] = (-1)^{|I|q + \theta(I,J)}[u] \sqcup [v],$$

where $\theta(I, J)$ is defined by $\theta(I, J) = \sum_{i \in I} \theta(i, J)$, and $\theta(i, J)$ is the number of elements $j \in J$, such that $j < i$.

In fact the formula given in [12, Proposition 3.2.10] (also [7, Theorem 10.1]) is not exactly the same as in Theorem 2.15. They showed that if $\sigma = (l_1, \ldots, l_p) \in \widetilde{C}^{p-1}(K_I)$ and $\tau = (m_1, \ldots, m_q) \in \widetilde{C}^{q-1}(K_J)$ are two cochain simplices with $l_1 < \cdots < l_p$, $m_1 < \cdots < m_q$, then

$$\sigma * \tau = \begin{cases} (-1)^\zeta \sigma \uplus \tau & \text{if } I \cap J = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$\zeta = \theta(\sigma, I) + \theta(\tau, J) + \theta(\sigma \cup \tau, I \cup J) + \theta(I \setminus \sigma, J \setminus \tau),$$

and $\sigma \uplus \tau$ is the cochain simplex of $\widetilde{C}^{p+q-1}(K_I * K_J)$ corresponding to $\sigma \sqcup \tau$ written in increasing order. Since ζ depends on both I, J and σ, τ, it is not convenient to use it to describe the relation between union product and star product. Now we prove that ζ is equal to the sign given in Theorem 2.15.

Proof of Theorem 2.15. First note that

$$\theta(\sigma \cup \tau, I \cup J) = \theta(\sigma, I) + \theta(\sigma, J) + \theta(\tau, I) + \theta(\tau, J),$$

and

$$\theta(I \setminus \sigma, J \setminus \tau) = \theta(I \setminus \sigma, J) - \theta(I \setminus \sigma, \tau).$$

Then

$$\zeta \equiv \theta(I, J) + \theta(\tau, I) + \theta(I \setminus \sigma, \tau) \mod 2$$

On the other hand, it is easy to see that

$$\sigma \uplus \tau = (-1)^{\theta(\sigma,\tau)} \sigma \sqcup \tau.$$

So if we can prove $\theta(\tau, I) + \theta(I, \tau) \equiv |I|q \mod 2$, then the formula holds. Note that

$$\theta(I, \tau) = \sum_{v \in I}(|\tau| - g(v, \tau)),$$

where $g(v, \tau)$ is the cardinality of the set $\{u \in \tau \mid u > v\}$. Let $f(u, v) = 1$ if $u > v$ and zero otherwise. Then we have

$$\theta(\tau, I) = \sum_{u \in \tau, v \in I} f(u, v) = \sum_{v \in I} g(v, \tau)$$

Combine the equations above and the fact that $|\tau| = q$, we get the desired equation. $\square$

2.4. **functorial properties of moment-angle complexes.** Let K be a simplicial complex on $[m]$, then $\mathcal{Z}_K^{[m]}$ can be seen as a subspace of the *unit polydisk* $(D^2)^m$ of $\mathbb{C}^m$, where

$$(D^2)^m = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \ i = 1, \ldots, m\}$$

A set map $\varphi : [l] \to [m]$ induces a map between polydisks

$$\psi : \prod_1^l (D^2)^l \to \prod_1^m (D^2)^m, \quad (x_1, \ldots, x_l) \to (y_1, \ldots, y_m),$$

where

$$y_j = \prod_{i \in \varphi^{(-1)}(j)} x_i \quad \text{for } j = 1, \ldots, m,$$

and set $y_j = 1$ if $\varphi^{(-1)}(j) = \varnothing$.

Proposition 2.16 ([36, Lemma 4.2]). *Let $\varphi : K_1 \to K_2$ be a simplicial map between simplicial complexes K_1 and K_2 on $[m_1]$ and $[m_2]$ respectively. Then there is a equivariant map $\varphi_{\mathcal{Z}} : \mathcal{Z}_{K_1}^{[m_1]} \to \mathcal{Z}_{K_2}^{[m_2]}$, which is the restriction of ψ (defined as above) to $\mathcal{Z}_{K_1}^{[m_1]}$.*

So there is a covariant functor $\mathcal{Z} : K \mapsto \mathcal{Z}_K^{[m]}$ from the category of finite simplicial complexes and simplicial maps to the category of spaces with torus actions and equivariant maps (the *moment-angle complex functor*).

Meanwhile, (the notation is as in Proposition 2.16) define a homomorphism $f : \mathbf{k}[m_2] \to \mathbf{k}[m_1]$ by

$$f(u_j) := \sum_{i \in \varphi^{(-1)}(j)} v_i.$$

Proposition 2.17 ([36, Proposition 2.2]). *f induces a homomorphism $\mathbf{k}(K_2) \to \mathbf{k}(K_1)$, and then induces a homomorphism*

$$\varphi_{\mathrm{Tor}}^* : \mathrm{Tor}_{\mathbf{k}[m_2]}^{*,*}(\mathbf{k}(K_2), \mathbf{k}) \to \mathrm{Tor}_{\mathbf{k}[m_1]}^{*,*}(\mathbf{k}(K_1), \mathbf{k}),$$

where φ_{Tor}^* is induced by the change of rings.

Thus we have a contravariant functor

$$\text{Tor-alg} : K \mapsto \text{Tor}^{*,\,*}_{\mathbf{k}[m]}(\mathbf{k}(K), \mathbf{k}).$$

from simplicial complexes to bigraded k-algebras.

On the other hand, Baskakov in [5] defined a homomorphism

$$\varphi^*_{\mathcal{H}} : \mathcal{H}^*_{[m_2]}(K_2) \to \mathcal{H}^*_{[m_1]}(K_1),$$

which is generated by the homomorphisms

$$\varphi^*_{I,J} : \widetilde{H}^*((K_2)_J) \to \widetilde{H}^*((K_1)_I) \ \text{ for all } I \subset [m_1], \ J \subset [m_2],$$

where $\varphi^*_{I,J} = \varphi^* \mid (K_1)_I : \widetilde{H}^*((K_2)_J) \to \widetilde{H}^*((K_1)_I)$ if $\varphi(I) = J$ and $|I| = |J|$, and $\varphi^*_{I,J} = 0$ in all other cases. In particular, if K_1 is a subcomplex of K_2, then $\varphi^*_{\mathcal{H}} = \bigoplus_{I \subset [m_1]} \varphi^*_I$, where $\varphi^*_I = \varphi^* \mid (K_1)_I : \widetilde{H}^*((K_2)_I) \to \widetilde{H}^*((K_1)_I)$. So we have another contravariant functor:

$$\mathcal{H} : K \mapsto \mathcal{H}^*_{[m]}(K).$$

Convention. We use the simplified notation φ^* to denote $\varphi^*_{\mathcal{H}}$ whenever it creates no confusion.

The relation between these functors is given by the following

Theorem 2.18 ([36, Theorem 4.7, Theorem 5.1]). *Let $H^* : X \to H^*(X; \mathbf{k})$ be the cohomology functor. Then there are natural equivalences induced by the isomorphisms in Theorem 2.12, that is*

$$H^* \circ \mathcal{Z} = \text{Tor-alg} = \mathcal{H}.$$

3. Gorenstein* complexes and Alexander duality

In commutative algebra, *Gorenstein rings* are an important class of rings because of their nice properties such as finite injective dimension and self-duality. In the language of homological algebra, a Gorenstein ring is a *Cohen-Macaulay ring of type one* (cf. [38, 8]). If the face ring (over a field $\mathbf{k}$) of a simplicial complex K is Gorenstein and K is not a cone, then it is called *Gorenstein* over* $\mathbf{k}$. The Gorenstein property has several topological and algebraic interpretations (some of which we list below) and is very important for both topological and combinatorial applications of the face rings.

The Koszul homology of a Gorenstein* ring behaves like the cohomology algebra of a manifold: it satisfies *Poincaré duality*. This fundamental result was proven by Avramov and Golod [1] for general Gorenstein rings. Here we state the graded version of their theorem in the case of face rings.

Definition 3.1. A finite dimensional graded $\mathbf{k}$-algebra $A = \bigoplus_{i=0}^d A^i$ ($\mathbf{k}$ is a field) will be called a *Poincaré algebra* if $\dim_{\mathbf{k}} A^d = 1$, and the $\mathbf{k}$-homomorphisms

$$A^i \to \text{Hom}_{\mathbf{k}}(A^{d-i}, A^d),$$
$$a \mapsto \phi_a, \quad \text{where } \phi_a(b) = ab$$

are isomorphisms for $0 \leqslant i \leqslant d$.

Theorem 3.2 ([1, Avramov-Golod], [12, Theorem 3.4.4])**.** *For a simplicial complex* K, *the face ring* $\mathbf{k}(K)$ *over a field* $\mathbf{k}$ *is a Gorenstein ring if and only if the algebra* $T^* = \bigoplus_{i=0}^{d} T^i$, *where* $T^i = \operatorname{Tor}_{\mathbf{k}[m]}^{-i}(\mathbf{k}(K), \mathbf{k})$, *is a Poincaré algebra.*

A theorem due to Stanley gives a purely topological characterisation of Gorenstien* complexes.

Theorem 3.3 ([38, Theorem 5.1])**.** *A simplicial complex* K *is Gorenstein* over a field* $\mathbf{k}$ *if and only if for any simplex* $\sigma \in K$ *(including* $\sigma = \varnothing$)

$$\widetilde{H}_i(\operatorname{link}_K \sigma; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } i = \dim(\operatorname{link}_K \sigma); \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently, $|K|$ *is a GHS over* $\mathbf{k}$.

The proof of Stanley's theorem is based on the calculation of the *local cohomology* of face rings, which was first given by Hochster for general simplicial complexes. Here we will give a more topological proof of Theorem 3.3 by using the connection between Poincaré duality of moment-angle manifolds and Alexander duality of GHS's.

Notice that a $\mathbf{k}$-orientable homology manifold over $\mathbf{k}$ clearly satisfies Poincaré duality with coefficients in $\mathbf{k}$. Since the proof of Alexander duality is a standard combination of Poincaré duality and excision, together with the homology information of S^n, then it can be naturally generalized to GHS's. Moreover, the Alexander duality has a more explicit description for the triangulation of a GHS:

Theorem 3.4 (Alexander duality)**.** *Let* K *be a simplicial complex with vertex set* $[m]$. *Then* $|K|$ *is a GHS over* $\mathbf{k}$ *of dimension* n *if and only if*

$$\widetilde{H}^i(K_I; \mathbf{k}) \cong \widetilde{H}_{n-i-1}(K_{[m] \setminus I}; \mathbf{k}) \quad \text{for all } I \subset [m], \ -1 \leq i \leq n.$$

Moreover, if the above isomorphisms hold, then they are induced by the excision product map:

$$D : \widetilde{H}^i(K_I; \mathbf{k}) \xrightarrow{[K] \sqcap} \widetilde{H}_{n-i-1}(K_{[m] \setminus I}; \mathbf{k}),$$

where $[K] \in \widetilde{H}_n(K; \mathbf{k})$ *is a fundamental class for* K.

Proof. The 'only if' part of the theorem follows from the Alexander duality and the fact that the homotopy equivalence $|K| - |K_I| \simeq |K_{[m] \setminus I}|$ holds for any simplicial complex K on $[m]$ and any $I \subset [m]$.

For the 'if' part, first we show by induction on the vertex number of σ that if $\sigma \cup I \in K$ for some $I \neq \varnothing$ with $I \cap \sigma = \varnothing$ then $\widetilde{H}_*(\operatorname{link}_{K_{\widehat{I}}} \sigma) = 0$, where $\widehat{I} = [m] \setminus I$. The case that $\sigma = \varnothing$ is true since $\widetilde{H}_*(\operatorname{link}_{K_{\widehat{I}}} \varnothing) = \widetilde{H}_*(K_{\widehat{I}}) = \widetilde{H}^*(K_I) = 0$ by duality in the theorem. For the induction step, suppose $\sigma = (v_1, \ldots, v_k), \sigma' = (v_2, \ldots, v_k)$ and $I' = I \cup \{v_1\}$, then it is easy to verify that

$$\operatorname{link}_{K_{\widehat{I}}} \sigma' = \operatorname{link}_{K_{\widehat{I'}}} \sigma' \cup_{\operatorname{link}_{K_{\widehat{I}}} \sigma} (\{v_1\} * \operatorname{link}_{K_{\widehat{I}}} \sigma).$$

Consider the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial} \widetilde{H}_i(\mathrm{link}_{K_{\hat{I}}}\sigma) \to \widetilde{H}_i(\mathrm{link}_{K_{\hat{I'}}}\sigma') \oplus \widetilde{H}_i(\{v_1\} * \mathrm{link}_{K_{\hat{I}}}\sigma) \to \widetilde{H}_i(\mathrm{link}_{K_{\hat{I}}}\sigma') \xrightarrow{\partial}$$
$$\widetilde{H}_{i-1}(\mathrm{link}_{K_{\hat{I}}}\sigma) \to \cdots$$

Since $\widetilde{H}_*(\{v_1\} * \mathrm{link}_{K'}\sigma) = 0$, and $\widetilde{H}_*(\mathrm{link}_{K_{\hat{I'}}}\sigma') = H_*(\mathrm{link}_{K_{\hat{I}}}\sigma') = 0$ $(\sigma' \cup I', \sigma' \cup I \in K)$ by induction, then $\widetilde{H}_*(\mathrm{link}_{K'}\sigma) = 0$, finishing the induction step. Next, we show that $\widetilde{H}_*(\mathrm{link}_K\sigma; \mathbf{k}) \cong \widetilde{H}_*(S^{n-|\sigma|}; \mathbf{k})$ for each $\sigma \in K$ by induction on the vertex number of $\sigma \in K$. The case $\sigma = \varnothing$ is clearly true. For the induction step, suppose $\sigma = (v_1, \ldots, v_k), \sigma' = (v_2, \ldots, v_k)$ and $K' = K_{[m]\setminus\{v_1\}}$ then as above

$$\mathrm{link}_K\sigma' = \mathrm{link}_{K'}\sigma' \cup_{\mathrm{link}_K\sigma} (\{v_1\} * \mathrm{link}_K\sigma).$$

Now consider the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\partial} \widetilde{H}_i(\mathrm{link}_K\sigma) \to \widetilde{H}_i(\mathrm{link}_{K'}\sigma') \oplus \widetilde{H}_i(\{v_1\} * \mathrm{link}_K\sigma) \to \widetilde{H}_i(\mathrm{link}_K\sigma') \xrightarrow{\partial}$$
$$\widetilde{H}_{i-1}(\mathrm{link}_K\sigma) \to \cdots$$

By induction, $\widetilde{H}_*(\mathrm{link}_K\sigma') = \widetilde{H}_{n-|\sigma'|}(\mathrm{link}_K\sigma') = \mathbf{k}$. $\widetilde{H}_*(\{v_1\} * \mathrm{link}_K\sigma) = 0$ is clear. The preceding argument shows that $\widetilde{H}_*(\mathrm{link}_{K'}\sigma') = 0$. Thus the exactness of the Mayer-Vietoris sequence implies that $\widetilde{H}_*(\mathrm{link}_K\sigma) \cong \widetilde{H}_*(S^{n-|\sigma|})$, finishing the induction step. It remains to show that K is pure of dimension n (this implies $\dim \mathrm{link}_K\sigma = n - |\sigma|$). To see this, let σ be a facet, then $\mathrm{link}_K\sigma = \varnothing$, so $\widetilde{H}_*(\varnothing) = \widetilde{H}_*(S^{n-|\sigma|})$ gives the desired result $\dim \sigma = n$.

To show that the Alexander duality is induced by the excision product, we first note that when the Alexander duality holds, $|K|$ is a GHS over $\mathbf{k}$. Hence from Theorem 2.10 (it can be extended in an obvious fashion to the case that $|K|$ is a GHS over $\mathbf{k}$), we have that $\mathcal{Z}_K$ is a homology $(m+n+1)$-manifold over $\mathbf{k}$. Let $[\mathcal{Z}_K]$ be a fundamental class of $\mathcal{Z}_K$ over $\mathbf{k}$. Then by Poincaré duality, there are isomorphisms

$$H^i(\mathcal{Z}_K; \mathbf{k}) \xrightarrow{[\mathcal{Z}_K]\frown} H_{m+n+1-i}(\mathcal{Z}_K; \mathbf{k}), \quad 0 \le i \le m+n+1.$$

Since the cap product for $\mathcal{Z}_k$ corresponds (up to sign) to the excision product for K, and the top homology class $[\mathcal{Z}_k]$ corresponds to the top homology class $[K]$, this is equivalent to saying that the maps

$$D : \widetilde{H}^i(K_I; \mathbf{k}) \xrightarrow{[K]\frown} \widetilde{H}_{n-i-1}(K_{[m]\setminus I}; \mathbf{k}), \quad I \subset [m], \; -1 \le i \le n.$$

are isomorphisms. The proof is finished. $\qquad\square$

Actually Theorem 3.4 gives another equivalent condition for K to be Gorenstein*. Meanwhile we can make use of it together with Theorem 3.2 to give a topological proof for Theorem 3.3:

Proof of Theorem 3.3. We may assume the vertex number of K is m and $\dim K = n - 1$. For the 'if' part, note that $\dim_{\mathbf{k}} T^{m-n} \ge 1$ by Theorem 2.12. We assert that

$\dim_{\mathbf{k}} T^{m-n} = 1$ and $T^i = 0$ for $i > m-n$. Otherwise, there exists $I \subsetneq [m]$ and $i \geq m-n$ such that $\widetilde{H}^{|I|-i-1}(K_I) \neq 0$. Hence from Alexander duality of K, $\widetilde{H}_{n+i-|I|-1}(K_{[m]\setminus I}) \neq 0$. Since $i \geq m-n$, we have $n+i-|I|-1 \geq m-|I|-1$. However a simplicial complex with $m-|I|$ vertices is either a simplex or of dimension less than $m-|I|-1$. Therefore, $\widetilde{H}_{n+i-|I|-1}(K_{[m]\setminus I}) = 0$, a contradiction. On the other hand, Theorem 3.4 says that the Alexander dual of K is induced by the excision product map $[K]\sqcap$. It is equivalent to saying that the bilinear form $T^i \times T^{m-n-i} \to T^{m-n}$ is nondegenerate because of the relation between union and excision product. Thus $T^* = \bigoplus_{i=0}^{m-n} T^i$ is a Poincaré algebra. Theorem 3.2 gives the desired result.

To prove the 'only if' part, first note that $T^d = \mathbf{k}$ for some $d > 0$ by Theorem 3.2. We assert that $T^d = \widetilde{H}^{m-d-1}(K)$. If this is not true, then $T^d = \widetilde{H}^{|I|-d-1}(K_I)$ for some $I \neq [m]$ by Theorem 2.12. This implies that $\widetilde{H}^*(K_J) = 0$ for any $J \not\subset I$ since the bilinear form $T^i \times T^{d-i} \to T^d$ is nondegenerate. It turns out that any missing face of K belongs to I, and it is equivalent to saying that $K = \Delta^{m-|I|-1} * K_I$, which is contradict to the assumption that K is not a cone. Now, by the Poincaré duality of T^* and the isomorphism $T^* \cong \mathcal{H}^*(K;\mathbf{k})$, we immediately have that $\widetilde{H}^i(K_I;\mathbf{k}) \cong \widetilde{H}^{m-d-i-2}(K_{[m]\setminus I};\mathbf{k})$ for $-1 \leq i \leq m-d-1$, $I \subset [m]$. On the other hand, $\widetilde{H}^*(K_{[m]\setminus I};\mathbf{k}) \cong \widetilde{H}_*(K_{[m]\setminus I};\mathbf{k})$ for the assumption that $\mathbf{k}$ is a field. Thus it follows from Theorem 3.4 that $|K|$ is a GHS over $\mathbf{k}$ of dimension $m-d-1$ (so $n = d-m-1$), finishing the proof. $\square$

For convention, we simply call a simplicial complex to be *Gorenstein** if the formula in Theorem 3.3 holds for $\mathbb{Z}$ coefficient. The following two results are easily deduced from the topological characterisation of Gorenstien* complexes.

Proposition 3.5 ([35, Theorem 6.1]). *If a simplicial complex K is Gorenstein* over $\mathbf{k}$, then K is pure, and for any simplex $\sigma \in K$, $\mathrm{link}_K \sigma$ is also Gorenstein* over $\mathbf{k}$.*

Proposition 3.6 ([33, Corollary 4.8]). *Let K_1 and K_2 be two Gorenstein* complexes over $\mathbf{k}$ with the same dimension. Then $K_1 \# K_2$ is also Gorenstein* over $\mathbf{k}$.*

4. $H^*(\mathcal{Z}_{K_1 \# K_2})$ FOR GORENSTEIN* COMPLEXES

As we have seen in Definition 2.3, for two pure simplicial complexes K_1 and K_2 with the same dimension, the combinatorial equivalent classes in $\mathcal{C}(K_1 \# K_2)$ are not unique in general. Nevertheless, if K_1 and K_2 are both Gorenstein*, we can give an universal formula for $H^*(\mathcal{Z}_{K_1 \# K_2})$ in terms of $H^*(\mathcal{Z}_{K_1})$ and $H^*(\mathcal{Z}_{K_2})$. Firstly, we introduce an algebraic operation on graded algebras.

Construction 4.1. For a finitely graded commutative $\mathbf{k}$-algebra $A = \bigoplus_{i=0}^{d} A^i$, define a operation G on A, called the *algebraic gyration of A*, as

$$G(A) = (A \otimes \Lambda_{\mathbf{k}}[v])/(A^d \otimes 1), \ \deg(v) = 1.$$

Let G^m denote the composition of m G's. Then it is easily verified that

$$G^m(A) = (A \otimes \Lambda_{\mathbf{k}}[v_1, \ldots, v_m])/(\bigoplus_{I \neq [m]} A^d \otimes v_I),$$

where v_I is a simplified notation for $\prod_{i \in I} v_i$.

Theorem 4.2. *Let K_1 and K_2 be two $(n-1)$-dimensional $(n \geq 2)$ Grenstein* complex over $\mathbf{k}$ with m_1 and m_2 vertices respectively, and let $K = K_1 \# K_2$. Then the reduced cohomology ring of $\mathcal{Z}_K$ is given by the isomorphism*

$$\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k}) \cong \mathcal{R}(K_1, K_2; \mathbf{k})/\mathcal{I}(K_1, K_2; \mathbf{k}),$$

where

$$\mathcal{R}(K_1, K_2; \mathbf{k}) = G^{m_2-n}(\widetilde{H}^*(\mathcal{Z}_{K_1}; \mathbf{k})) \times G^{m_1-n}(\widetilde{H}^*(\mathcal{Z}_{K_2}; \mathbf{k})) \times \widetilde{H}^*(M; \mathbf{k}),$$

$$M = \overset{m_1+m_2-2n}{\underset{i=2}{\sharp}} \lambda(i)(S^{i+1} \times S^{m_1+m_2-i-1}),$$

$$\lambda(i) = \binom{m_1+m_2-2n}{i} - \binom{m_1-n}{i} - \binom{m_2-n}{i},$$

$\mathcal{I}(K_1, K_2; \mathbf{k})$ is an ideal of $\mathcal{R}(K_1, K_2; \mathbf{k})$ generated by

$$([Z_1], 0, 0) - (0, 0, [M]) \ \text{and} \ (0, [Z_2], 0) - (0, 0, [M]),$$

where $[Z_1]$ (resp. $[Z_2]$, $[M]$) is the top dimensional generator of $G^{m_2-n}(\widetilde{H}^*(\mathcal{Z}_{K_1}; \mathbf{k}))$ (resp. $G^{m_1-n}(\widetilde{H}^*(\mathcal{Z}_{K_2}; \mathbf{k}))$, $\widetilde{H}^*(M; \mathbf{k})$).

Before proving this main theorem, we first give several more general results about the cohomology of moment-angle complexes.

Proposition 4.3. *Suppose we are given two simplices Δ^{m_1-1} and Δ^{m_2-1}. Let $L = \Delta^{m_1-1} \cup_\sigma \Delta^{m_2-1}$ denote the simplicial complex obtained from Δ^{m_1-1} and Δ^{m_2-1} by gluing along a common simplex σ of dimension $n-1$. Then*

$$\widetilde{H}^*(\mathcal{Z}_L; \mathbf{k}) \cong \widetilde{H}^*(\bigvee_{i \geq 2} \lambda_i S^{i+1}; \mathbf{k}),$$

where

$$\lambda(i) = \binom{m_1+m_2-2n}{i} - \binom{m_1-n}{i} - \binom{m_2-n}{i}.$$

Proof. Set $\mathcal{S}_1$ and $\mathcal{S}_2$ are the vertex sets of Δ^{m_1-1} and Δ^{m_2-1} respectively. From the construction of L we get that

$$\widetilde{H}^i(L_I; \mathbf{k}) = \begin{cases} \mathbf{k} & \text{if } i = 0 \text{ and } I \cap \sigma = \varnothing, \ I \cap \mathcal{S}_1 \neq \varnothing, \ I \cap \mathcal{S}_2 \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

So the additive isomorphism follows from theorem 2.12 immediately. To verify the ring structure, given $[u] \in \widetilde{H}^0(L_{I_1})$ and $[v] \in \widetilde{H}^0(L_{I_2})$ with $I = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, then $[u] * [v] \in \widetilde{H}^1(L_I) = 0$. This proves the proposition. $\square$

Proposition 4.4. *Let K_1 be a simplicial complex with m_1 vertices, and let $L_1 = K_1 \cup_\sigma \Delta^{m_2-1}$ denote the simplicial complex obtained from K_1 and Δ^{m_2-1} by gluing along a common simplex σ, and let $L = \Delta^{m_1-1} \cup_\sigma \Delta^{m_2-1}$. Then*

$$\widetilde{H}^*(\mathcal{Z}_{L_1}; \mathbf{k}) \cong (\widetilde{H}^*(\mathcal{Z}_{K_1}; \mathbf{k}) \otimes \Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2-|\sigma|}]) \times \widetilde{H}^*(\mathcal{Z}_L; \mathbf{k})$$

where $\Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2-|\sigma|}]$ denotes the graded exterior algebra over $\mathbf{k}$ with $m_2 - |\sigma|$ generators; $\deg(v_i) = 1$ for $1 \leq i \leq m_2 - |\sigma|$.

Proof. The coefficient ring $\mathbf{k}$ will play no special role in the argument so we shall omit it from the notation. Suppose $m = m_1 + m_2 - |\sigma|$. Set

$$[m_1] = \{u_1, \ldots, u_{m_1-|\sigma|}, w_1, \ldots, w_{|\sigma|}\},$$
$$[m_2] = \{v_1, \ldots, v_{m_2-|\sigma|}, w_1, \ldots, w_{|\sigma|}\}, \text{ and}$$
$$[m] = \{u_1, \ldots, u_{m_1-|\sigma|}, w_1, \ldots, w_{|\sigma|}, v_1, \ldots, v_{m_2-|\sigma|}\}$$

be the vertex sets of K_1, Δ^{m_2-1} and L_1 respectively. Consider simplicial inclusions: $K_1 \xhookrightarrow{i_1} L_1 \xhookrightarrow{i_2} L$, which induce inclusions $\mathcal{Z}_{K_1}^{[m]} \xhookrightarrow{\phi_1} \mathcal{Z}_{L_1} \xhookrightarrow{\phi_2} \mathcal{Z}_L$. By Theorem 2.18, we have a commutative diagram

$$
\begin{array}{ccccc}
\widetilde{H}^*(\mathcal{Z}_L) & \xrightarrow{\phi_2^*} & \widetilde{H}^*(\mathcal{Z}_{L_1}) & \xrightarrow{\phi_1^*} & \widetilde{H}^*(\mathcal{Z}_{K_1}^{[m]}) \\
\cong \downarrow & & \cong \downarrow & & \cong \downarrow \\
\widetilde{\mathcal{H}}^*(L) & \xrightarrow{i_2^*} & \widetilde{\mathcal{H}}^*(L_1) & \xrightarrow{i_1^*} & \widetilde{\mathcal{H}}^*_{[m]}(K_1)
\end{array}
$$

From the construction of L and L_1 it is easy to see that for any $I \subseteq [m]$, $I \neq \varnothing$

$$
\begin{aligned}
(L_1)_I &\cong L_I \simeq pt, \quad \text{if } I \cap [m_1] = \varnothing, \\
(L_1)_I &\simeq (K_1)_I \vee L_I, \quad \text{if } I \cap [m_1] \neq \varnothing,
\end{aligned}
\tag{4.1}
$$

and in the second case we have a commutative diagram

$$
\begin{array}{ccccc}
(K_1)_I & \xrightarrow{(i_1)_I} & (L_1)_I & \xrightarrow{(i_2)_I} & L_I \\
\| & & \simeq \downarrow & & \| \\
(K_1)_I & \xrightarrow{j_I} & (K_1)_I \vee L_I & \xrightarrow{p_I} & L_I
\end{array}
$$

where j_I and p_I are the natural inclusion and projection maps respectively. Therefore $\widetilde{\mathcal{H}}^*(L_1)$ additively splits as $A_1 \oplus A_2$, where

$$A_1 \cong i_1^*(A_1) = \operatorname{Im} i_1^*; \quad A_2 = \operatorname{Im} i_2^* = \operatorname{Ker} i_1^* \cong \widetilde{\mathcal{H}}^*(L).$$

Actually, this splitting is also multiplicative. Since A_2 is an ideal of $\widetilde{\mathcal{H}}^*(L_1)$, we need only to verify that A_1 is a subalgebra and for any $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$, $\alpha_1 * \alpha_2 = 0$. Given two elements $\alpha, \alpha' \in A_1$, then $\alpha * \alpha' \in \bigoplus_{i \geq 2} \widetilde{\mathcal{H}}^i(L_1)$. However, from Proposition 4.3, we have $\widetilde{\mathcal{H}}^*(L) = \widetilde{\mathcal{H}}^1(L)$, so $\alpha * \alpha' \in A_1$. A_1 is an algebra. Meanwhile, for any $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$, $\alpha_1 * \alpha_2 = 0 \in A_2$ for the same dimension reason. Thus $\widetilde{\mathcal{H}}^*(L_1) = A_1 \times A_2$.

According to Theorem 2.15, a straightforward calculation shows that

(1) In $\mathcal{H}^*_{[m]}(K_1)$, the subalgebra $\bigoplus\limits_{I \cap [m_1] = \varnothing} \widetilde{H}^*((K_1)_I) \cong \Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2 - |\sigma|}]$. Each generator v_i of the exterior algebra corresponds to a generator of $\widetilde{H}^{-1}((K_1)_{\{v_i\}}) = \widetilde{H}^{-1}(\varnothing) = \mathbf{k}$.

(2) For $I = I_1 \cup I_2$ with $I_1 \subseteq [m_1]$, $I_2 \cap [m_1] = \varnothing$, the homomorphism

$$\widetilde{H}^*((K_1)_{I_1}) \otimes \widetilde{H}^*((K_1)_{I_2}) \to \widetilde{H}^*((K_1)_I)$$
$$\alpha \otimes \beta \mapsto \alpha * \beta$$

is an isomorphism.

Therefore we have an isomorphism

$$\bigoplus_{I \subseteq [m]} \widetilde{H}^*((K_1)_I) \cong \Big(\bigoplus_{I \subseteq [m_1]} \widetilde{H}^*((K_1)_I) \Big) \otimes \Big(\bigoplus_{I \cap [m_1] = \varnothing} \widetilde{H}^*((K_1)_I) \Big)$$

From formula (4.1), it is easy to see that

$$\operatorname{Im} i_1^* = \bigoplus_{I \cap [m_1] \neq \varnothing} \widetilde{H}^*((K_1)_I) \cong \Big(\bigoplus_{\substack{I \subseteq [m_1] \\ I \neq \varnothing}} \widetilde{H}^*((K_1)_I) \Big) \otimes \Big(\bigoplus_{I \cap [m_1] = \varnothing} \widetilde{H}^*((K_1)_I) \Big)$$

Then the proposition follows at once. $\qquad\qquad\square$

Proposition 4.5. *Let K_1 and K_2 be simplicial complexes with m_1 and m_2 vertices respectively. Let $K = K_1 \cup_\sigma K_2$ denote a simplicial complex obtained from K_1 and K_2 by gluing along a common simplex σ, and let $L = \Delta^{m_1 - 1} \cup_\sigma \Delta^{m_2 - 1}$. Then*

$$\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k}) \cong (\widetilde{H}^*(\mathcal{Z}_{K_1}; \mathbf{k}) \otimes \Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2 - |\sigma|}])$$
$$\times (\widetilde{H}^*(\mathcal{Z}_{K_2}; \mathbf{k}) \otimes \Lambda_{\mathbf{k}}[u_1, \ldots, u_{m_1 - |\sigma|}]) \times \widetilde{H}^*(\mathcal{Z}_L; \mathbf{k})$$

Proof. We use the same notations $[m_1]$, $[m_2]$ and $[m]$ as in the proof of Proposition 4.4 to denote the vertex sets of K_1, K_2 and K respectively. Let $L_1 = K_1 \cup_\sigma \Delta^{m_2 - 1}$, $L_2 = \Delta^{m_1 - 1} \cup_\sigma K_2$, then we have a commutative diagram of simplicial inclusions

which induces a commutative diagram of algebras

$$
\begin{array}{ccccc}
\widetilde{\mathcal{H}}^*_{[m]}(K_1) & & \widetilde{\mathcal{H}}^*(L_2) & & \\
& \searrow^{i_1^*} \quad \swarrow^{j_2^*} & & \nwarrow^{l_2^*} & \\
& & \widetilde{\mathcal{H}}^*(K) & & \widetilde{\mathcal{H}}^*(L) \\
& \nearrow^{i_2^*} \quad \nwarrow^{j_1^*} & & \swarrow^{l_1^*} & \\
\widetilde{\mathcal{H}}^*_{[m]}(K_2) & & \widetilde{\mathcal{H}}^*(L_1) & &
\end{array}
$$

By the construction of these simplicial complexes, for any $I \subseteq [m]$ we have

$$
\begin{aligned}
K_I &\cong (L_1)_I \cong (K_1)_I \quad \text{if } I \cap [m_2] = \varnothing, \\
K_I &\cong (L_2)_I \cong (K_2)_I \quad \text{if } I \cap [m_1] = \varnothing, \\
K_I &\simeq (K_1)_I \vee (L_2)_I \simeq (K_2)_I \vee (L_1)_I \quad \text{otherwise.}
\end{aligned} \tag{4.2}
$$

So $\widetilde{\mathcal{H}}^*(K)$ additively splits as $A_1 \oplus A_2$ (also as $B_1 \oplus B_2$), where

$$
A_1 \cong \operatorname{Im} i_1^* = i_1^*(A_1) \ (B_1 \cong \operatorname{Im} i_2^* = i_2^*(B_1)), \text{ and}
$$

$$
A_2 = \operatorname{Im} j_2^* = \operatorname{Ker} i_1^* \cong \widetilde{\mathcal{H}}^*(L_2) \ (B_2 = \operatorname{Im} j_1^* = \operatorname{Ker} i_2^* \cong \widetilde{\mathcal{H}}^*(L_1)).
$$

From formula (4.2), we also have $\operatorname{Im} i_1^* = \operatorname{Im} (j_1 i_1)^*$. From the proof of Proposition 4.4, we know that $\widetilde{\mathcal{H}}^*(L_1)$ multiplicatively splits as $A_1' \oplus A_2'$, where

$$
A_1' \cong \operatorname{Im} (j_1 i_1)^* \cong \widetilde{H}^*(\mathcal{Z}_{K_1}) \otimes \Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2 - |\sigma|}],
$$

$$
A_2' = \operatorname{Im} l_1^* \cong \widetilde{\mathcal{H}}^*(L).
$$

So we can take $A_1 = j_1^*(A_1')$.

Now we will show that the splitting $\widetilde{\mathcal{H}}^*(K) = A_1 \oplus A_2$ is a multiplicative splitting. It is clear that A_1 is a subalgebra and A_2 is an ideal. For any $\alpha_1 \in A_1$, $\alpha_2 \in A_2$, $\alpha_1 * \alpha_2 \in A_2$ and $i_2^*(\alpha_1 * \alpha_2) = i_2^*(\alpha_1) * i_2^*(\alpha_2) = 0$ $(A_1 \subseteq \operatorname{Im} j_1^* = \operatorname{Ker} i_2^*)$. Thus $\alpha_1 * \alpha_2 \in \operatorname{Ker} i_2^* = \operatorname{Im} j_1^*$. Note that $\operatorname{Im} j_1^* = j_1^*(A_1') \oplus j_1^*(A_2') = A_1 \oplus j_1^*(A_2')$. So $\alpha_1 * \alpha_2 \in j_1^*(A_2')$. By Proposition 4.3, the elements of A_2' all come from $\widetilde{\mathcal{H}}^1(L_1)$, but $\alpha_1 * \alpha_2 \in \bigoplus_{i \geq 2} \widetilde{\mathcal{H}}^i(K)$. It follows that $\alpha_1 * \alpha_2 = 0$. Therefore the above splitting is multiplicative, and the result follows. $\square$

Now let us complete the proof of Theorem 4.2.

Proof of Theorem 4.2. The cohomology groups with coefficients in $\mathbf{k}$ will be implicit throughout the proof. We use the notation $[m_1]$, $[m_2]$ and $[m]$, as in the proof of Proposition 4.5, to denote the vertex sets of K_1, K_2 and K respectively. Suppose σ is the selected common facet of K_1 and K_2 corresponding to K. Set $K' = K_1 \cup_\sigma K_2$, then there is a simplicial inclusion $i : K \hookrightarrow K'$, which induces a inclusion $\phi : \mathcal{Z}_K \hookrightarrow \mathcal{Z}_{K'}$. Consider the homomorphisms of cohomology groups

$$
i_I^* : \widetilde{H}^j(K_I') \to \widetilde{H}^j(K_I), \ I \subseteq [m], \ -1 \leq j \leq n - 1.
$$

We will analyze i_I^* in five cases:

(i) $\sigma \nsubseteq I$. In this case $K_I' = K_I$, so i_I^* is an isomorphism for $-1 \leq j \leq n - 1$.

(ii) $\sigma \subseteq I$ and $[m_1] \nsubseteq I$, $[m_2] \nsubseteq I$. In this case K_I is a proper subcomplex of K_I', and $K_I' \simeq (K_1)_I \vee (K_2)_I$. Consider the long exact sequence of cohomology groups

$$\cdots \to \widetilde{H}^{j-1}(K_I) \xrightarrow{\delta} \widetilde{H}^j(K_I', K_I) \to \widetilde{H}^j(K_I') \xrightarrow{i_I^*} \widetilde{H}^j(K_I) \xrightarrow{\delta} \cdots$$

By excision theorem

$$\widetilde{H}^j(K_I', K_I) \cong \widetilde{H}^j(\Delta^{n-1}, \partial\Delta^{n-1}) = \begin{cases} \mathbf{k} & \text{if } j = n - 1; \\ 0 & \text{otherwise.} \end{cases}$$

$\widetilde{H}^{n-1}(K_I') = \widetilde{H}^{n-1}(K_I) = 0$ by Theorem 3.4. Therefore from the long exact sequence above, we have that i_I^* is an isomorphism for $j \neq n - 2$ and $\widetilde{H}^{n-2}(K_I)$ splits as $\text{Im}\, i_I^* \oplus \mathbf{k}$.

(iii) $[m_1] \subseteq I$, $[m_2] \nsubseteq I$. The only difference to (ii) is that $\widetilde{H}^{n-1}(K_I') \cong \mathbf{k}$ in this case. A similar analysis shows that i_I^* is an isomorphism for $j < n - 1$ and trivial for $j = n - 1$. From the proof of Proposition 4.5, a generator of $\widetilde{H}^{n-1}(K_I') \cong \mathbf{k}$ corresponds to $[\mathcal{Z}_{K_1}] \otimes v_{I \setminus [m_1]}$ in the formula of Proposition 4.3, where $[\mathcal{Z}_{K_1}]$ is a generator of $\widetilde{H}^{m_1+n}(\mathcal{Z}_{K_1}) \cong \mathbf{k}$.

(iv) $[m_2] \subseteq I$, $[m_1] \nsubseteq I$. i_I^* is exactly the same as in (iii). In this case a generator of $\widetilde{H}^{n-1}(K_I')$ corresponds to $[\mathcal{Z}_{K_2}] \otimes u_{I \setminus [m_2]}$ in the formula of Proposition 4.3, where $[\mathcal{Z}_{K_2}]$ is a generator of $\widetilde{H}^{m_2+n}(\mathcal{Z}_{K_2}) \cong \mathbf{k}$.

(v) $I = [m]$. In this case

$$\widetilde{H}^{n-1}(K_I') = \widetilde{H}^{n-1}(K') \cong \mathbf{k} \oplus \mathbf{k}, \ \widetilde{H}^{n-1}(K_I) = \widetilde{H}^{n-1}(K) \cong \mathbf{k},$$

$$\widetilde{H}^j(K_I') = \widetilde{H}^j(K_I) = 0 \ \text{ for } \ j < n - 1.$$

Let ξ_1 and ξ_2 be two generators of $\widetilde{H}^{n-1}(K')$ corresponding respectively to $[\mathcal{Z}_{K_1}] \otimes v_{[m_2-n]}$ and $[\mathcal{Z}_{K_2}] \otimes u_{[m_1-n]}$ in the formula of Proposition 4.3. It is easy to see that $i^*(\xi_1) = i^*(\xi_2)$.

Combining the above arguments, we have that as $\mathbf{k}$-module

$$\widetilde{H}^*(\mathcal{Z}_K) = \phi^*(\widetilde{H}^*(\mathcal{Z}_{K'})) \oplus B,$$

where

$$B \cong \widetilde{H}^*\Big(\bigvee_{I \subseteq \mathcal{M}} S_I^{|I|+n-1}\Big), \quad \mathcal{M} = \{I \in [m] : \sigma \subseteq I, [m_1] \nsubseteq I, [m_2] \nsubseteq I\}.$$

Each sphere summand $S_I^{|I|+n-1}$ above corresponds to a $\mathbf{k}$ direct summand in $\widetilde{H}^{n-2}(K_I)$ (see case (ii)). Denote by β_I a generator of this $\mathbf{k}$ direct summand. On the other hand, by Proposition 4.5 $\widetilde{H}^*(\mathcal{Z}_{K'}) = A_1 \times A_2 \times A$ (as algebra), where

$$A_1 \cong \widetilde{H}^*(\mathcal{Z}_{K_1}) \otimes \Lambda_{\mathbf{k}}[v_1, \ldots, v_{m_2-|\sigma|}]; \quad A_2 \cong \widetilde{H}^*(\mathcal{Z}_{K_2}) \otimes \Lambda_{\mathbf{k}}[u_1, \ldots, u_{m_1-|\sigma|}];$$

$$A \cong \widetilde{H}^*(\bigvee_{J \in \mathcal{N}} S_J^{|J|+1}), \quad \mathcal{N} = \{I \subseteq [m] : I \cap \sigma = \varnothing, \, I \cap [m_1] \neq \varnothing, \, I \cap [m_2] \neq \varnothing\}.$$

Each sphere summand $S_J^{|J|+1}$ above corresponds to a $\mathbf{k}$ direct summand in $\widetilde{H}^0(K_J')$, denote by α_J a generator of this $\mathbf{k}$ direct summand. It is straightforward to see that $\mathcal{M}$ and $\mathcal{N}$ are in one-to-one correspondence: $I \mapsto [m] \setminus I$. The previous arguments imply that $\phi^*(A) \cong A$, and $\phi^*(A_1 \times A_2) \cong (A_1 \times A_2)/\mathcal{I}$ where $\mathcal{I}$ is a ideal generated by

$$([\mathcal{Z}_{K_1}] \otimes \alpha, 0) \text{ for } \alpha \in \bigoplus_{i=0}^{m_2-n-1} \Lambda_{\mathbf{k}}^i[v_1, \ldots, v_{m_2-n}] \text{ and}$$

$$(0, [\mathcal{Z}_{K_2}] \otimes \beta) \text{ for } \beta \in \bigoplus_{j=0}^{m_1-n-1} \Lambda_{\mathbf{k}}^j[u_1, \ldots, u_{m_1-n}] \text{ and}$$

$$([\mathcal{Z}_{K_1}] \otimes v_{[m_2-n]}, 0) - (0, [\mathcal{Z}_{K_2}] \otimes u_{[m_1-n]}).$$

An easy observation shows that the isomorphism in the theorem holds for $\mathbf{k}$-module homomorphism. Now let us complete the proof by verifying the ring structure of $\widetilde{H}^*(\mathcal{Z}_K)$. $\phi^*(\widetilde{H}^*(\mathcal{Z}_{K'}))$ is clearly an algebra. For any two generators $\beta_I, \beta_{I'} \in B$, $\varnothing \neq \sigma \subset I \cap I'$, so $\beta_I * \beta_{I'} = 0$ in $\widetilde{\mathcal{H}}^*(K)$. and then B is an algebra with trivial product structure. Thus we need only to verify the multiplication between B and $\phi^*(\widetilde{H}^*(\mathcal{Z}_{K'})) = i^*(\widetilde{\mathcal{H}}^*(K'))$.

If $\mathbf{k}$ is a field, then by Proposition 3.6 and Theorem 3.2, $\mathcal{H}^*(K)$ is a Poincaré $\mathbf{k}$-algebra. First we assert that $\beta_I \in B$ can be chosen properly such that $\beta_I * i^*(\alpha) = 0$ for any $\alpha \in A_1 \times A_2$. To see this, note first that if $\alpha \in A_1$ such that $\beta_I * i^*(\alpha) \neq 0$, then for the dimension reason, $\alpha \in A_1^1$ (define $A_1^i = A_1 \cap \widetilde{\mathcal{H}}^i(K')$). By arguments in case (v) $A_1^n \cong i^*(A_1^n) = \widetilde{\mathcal{H}}^n(K)$. Then for a chosen β_I there is a $\mathbf{k}$-homomorphism defined by

$$\phi_{\beta_I} : A_1^1 \to A_1^n,$$
$$a \mapsto (i^*)^{-1}(\beta_I * i^*(a)).$$

Since K_1 is Gorenstein* over $\mathbf{k}$, $\widetilde{\mathcal{H}}^*(K_1)$ (without concerning $\mathcal{H}^0(K_1)$) is a Poincaré algebra, and so is A_1. Thus there exists an element $a_1 \in A_1^{n-1}$, such that the $\mathbf{k}$-homomorphism

$$\phi_{a_1} : A_1^1 \to A_1^n,$$
$$a \mapsto a_1 * a$$

is equal to ϕ_{β_I}, and so

$$(\beta_I - i^*(a_1)) * i^*(a) = i^*(i^*)^{-1}(\beta_I * i^*(a)) - i^*(a_1 * a)$$
$$= i^*((i^*)^{-1}(\beta_I * i^*(a)) - a_1 * a) = 0$$

for any $a \in A_1$. Similarly there exists $a_2 \in A_2^{n-1}$, such that $(\beta_I - i^*(a_2)) * i^*(a) = 0$ for any $a \in A_2$. Replacing β_I by $\beta_I - i^*(a_1) - i^*(a_2)$ we get the desired generator. So we can make $B * i^*(A_1 \times A_2) = 0$.

It remains to verify the multiplication between $i^*(A)$ and B. For each generator $\beta_I \in B \subset \widetilde{\mathcal{H}}^{n-1}(K)$ and $\alpha_J \in A \subset \widetilde{\mathcal{H}}^1(K)$, we have that $\beta_I * i^*(\alpha_J) = 0$ if $J \neq [m] \setminus I$. This is because $\widetilde{H}^{n-1}(K_S) = 0$ if $S \neq [m]$. On the other hand, since $B * i^*(A_1 \times A_2) = 0$, then by the definition of Poincaré algebra $\beta_I * i^*(\alpha_{[m]\setminus I})$ is a generator of $\widetilde{\mathcal{H}}^n(K) = \mathbf{k}$. Thus by choosing the coefficients of $\{\beta_I\}$ and $\{\alpha_J\}$ properly, we have

$$\beta_I * i^*(\alpha_J) = \begin{cases} [\mathcal{Z}_{K_1}] \otimes v_{[m_2-n]} = [\mathcal{Z}_{K_2}] \otimes u_{[m_1-n]} & \text{if } I \cup J = [m],\ I \cap J = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Then the desired algebra isomorphism follows.

If $\mathbf{k} = \mathbb{Z}$. Write $K^{\{0\}} = K$, $K^{\{1\}} = K_1$ and $K^{\{2\}} = K_2$. If we prove that

$$\varphi : \widetilde{\mathcal{H}}^{n-1}(K^{\{i\}}) \to \operatorname{Hom}(\widetilde{\mathcal{H}}^1(K^{\{i\}}), \widetilde{\mathcal{H}}^n(K^{\{i\}}))$$

$$a \mapsto \phi_a, \qquad \phi_a(b) = a * b$$

is an isomorphism for $i = 0, 1, 2$, then the result can be proved in the same way as the field case. This is clearly true. Since when $K^{\{i\}}$ is Gorenstein*, $\mathcal{Z}_{K^{\{i\}}}$ is a manifold, thus satisfies Poincaré duality. Then φ is an isomorphism after factoring out the torsion. Note that $\widetilde{\mathcal{H}}^{n-1}(K^{\{i\}})$ and $\widetilde{\mathcal{H}}^1(K^{\{i\}})$ are always torsion free, the result follows. $\square$

Since when K_1 and K_2 are Gorenstein*, $\mathcal{Z}_{K_1}$, $\mathcal{Z}_{K_2}$ and $\mathcal{Z}_{K_1 \# K_2}$ are both manifolds, then we wish to obtain $\mathcal{Z}_{K_1 \# K_2}$ from $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ by making some geometrical operation on them. First we introduce the following operation on a manifold, which is studied in [23].

Definition 4.6. Let M^n be an n-manifold without boundary, and let M^n_{-1} be M^n minus an open ball $\overset{\circ}{D}^n$. The *gyration* $\mathcal{G}(M^n)$ of M^n is defined to be the manifold

$$\mathcal{G}(M^n) = \partial(M^n_{-1} \times D^2) = M^n_{-1} \times S^1 \cup S^{n-1} \times D^2.$$

Apparently, $\mathcal{G}(M^n)$ is obtained from the manifold $M^n \times S^1$ by an $(1, n)$-type surgery. A straightforward calculation shows the effect of gyration on cohomology.

Proposition 4.7. *Let M^n be a closed $\mathbf{k}$-orientable manifold of dimension n. Then there is a ring isomorohism*

$$\widetilde{H}^*(\mathcal{G}(M^n); \mathbf{k}) \cong G(\widetilde{H}^*(M^n; \mathbf{k})).$$

From Theorem 4.2 and Proposition 4.7, we immediately get the following Corollary.

Corollary 4.8. *Let K_1 and K_2 be two $n-1$ dimensional Gorenstein* complexes with m_1 and m_2 vertices respectively. Then $H^*(\mathcal{Z}_{K_1 \# K_2}; \mathbf{k}) \cong H^*(M; \mathbf{k})$, where*

$$M = \mathcal{G}^{m_2-n}(\mathcal{Z}_{K_1}) \# \mathcal{G}^{m_1-n}(\mathcal{Z}_{K_2}) \# \overset{m_1+m_2-2n}{\underset{i=2}{\sharp}} \lambda(i)(S^{i+1} \times S^{m_1+m_2-i-1}),$$

$$\lambda(i) = \binom{m_1+m_2-2n}{i} - \binom{m_1-n}{i} - \binom{m_2-n}{i},$$

and $\mathcal{G}^r(M)$ means iterating the gyration on a manifold M by r times.

Remark 4.9. Actually, if one of K_1 and K_2 is $\partial \Delta^n$ in the above corollary, then the cohomology ring isomorphism is induced by a homeomorphism of manifolds [32], i.e., for an $(n-1)$-dimensional Gorenstein* complex K with m vertices, we have a homeomorphism

$$\mathcal{Z}_{K\#\partial\Delta^n} \approx \mathcal{G}(\mathcal{Z}_K) \overset{m-n}{\underset{i=1}{\natural}} \binom{m-n}{i}(S^{i+2} \times S^{m+n-i-1}).$$

This is just the conjecture given by S. Gitler and S. López de Medrano [23].

For the general case we make the following

Conjecture 4.10. $\mathcal{Z}_{K_1\#K_2}$ is homeomorphic to

$$\mathcal{G}^{m_2-n}(\mathcal{Z}_{K_1})\#\mathcal{G}^{m_1-n}(\mathcal{Z}_{K_2})\# \overset{m_1+m_2-2n}{\underset{i=2}{\natural}}\lambda(i)(S^{i+1} \times S^{m_1+m_2-i-1}).$$

5. Change of $H^*(\mathcal{Z}_K)$ after a stellar subdivision on K

Definition 5.1. Let $\sigma \in K$ be a nonempty simplex of a simplicial complex K. The *stellar subdivision* of K at σ is obtained by replacing the star of σ by the cone over its boundary:

$$\mathrm{S}_\sigma K = (K \setminus \mathrm{star}_K\sigma) \cup \big(\mathrm{cone}(\partial\sigma * \mathrm{link}_K\sigma)\big).$$

If $\dim \sigma = 0$ then $\mathrm{S}_\sigma K = K$. Otherwise the complex $\mathrm{S}_\sigma K$ acquires an additional vertex (the vertex of the cone).

Definition 5.2. Two simplicial complexes K, L are called *stellar equivalent*, if there is a sequence of simplicial complexes $K = K_0, K_1, \ldots, K_n = L$ such that K_{i+1} is a stellar subdivision of K_i or K_i is a stellar subdivision of K_{i+1} for each i.

A classical result in PL theory says that

Theorem 5.3 ([31, Theorem 4.5]). *Two simplicial complexes are piecewise linearly homeomorphic if and only if they are stellar equivalent.*

Thus if we can find a formula (depends only on $\dim K$ and $\dim \sigma$) to describe the relation between $H^*(\mathcal{Z}_K)$ and $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ for any $\sigma \in K$, then we can actually get the cohomology of all moment-angle manifolds associated to PL-spheres. However, this is impossible, since the change of $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ from $H^*(\mathcal{Z}_K)$ does not only depend on $\dim \sigma$, but also on the position of σ in K in general. Actually, the calculation of $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ is very complicated even if the cohomology $H^*(\mathcal{Z}_K)$ is already known in general cases. Nevertheless, when σ satisfies some local conditions, we can get a simple description of $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$ by $H^*(\mathcal{Z}_K)$. That is the following

Theorem 5.4. *Let K be a Gorenstein* complex of dimension $n-1$ with m vertices, $\sigma \in K$ be a simplex of dimension $q < n-1$. Let $\mathcal{V}$ be the vertex set of $\mathrm{link}_K\sigma$, $s = |\mathcal{V}|+|\sigma|$. If for any $I \subset \mathcal{V}$, the inclusion map $\varphi_I : (\mathrm{link}_K\sigma)_I \to K_I$ is nullhomotopic, and if one of the following additional conditions are satisfied:*

(a) **k** *is a field.*

(b) $\mathbf{k} = \mathbb{Z}$ and $H^*(\mathcal{Z}_{\mathrm{link}_K\sigma})$ is torsion free.

Then the cohomology ring of $\mathcal{Z}_{\mathrm{S}_\sigma K}$ is given by the isomorphism

$$H^*(\mathcal{Z}_{\mathrm{S}_\sigma K}; \mathbf{k}) \cong H^*\big(\mathcal{G}(\mathcal{Z}_K) \# Y; \mathbf{k}\big),$$

where

$$Y = \underset{i+j\geq 1}{\sharp}\, f_i \cdot \binom{m-s}{j}(S^{i+j+2} \times S^{m+n-i-j-1}),$$

$$f_i = \mathrm{rank}\, H^i(\mathcal{Z}_{\mathrm{link}_K\sigma}).$$

Definition 5.5. A simplicial complex K is called *q-neighborly* if any q vertices of K span a simplex of K.

If K is a q-neighborly simplicial complex, then by definition the k-skeleton of K is homotopy equivalent to the wedge of S^k's for $k < q$. Hence $\pi_k(K) = 0$ for $k < q - 1$.

Corollary 5.6. *Let K be a Gorenstein* complex of dimension $n - 1$ with m vertices. If $\sigma \in K$ ($\dim \sigma < n - 1$) satisfies $\mathrm{link}_K\sigma \cong \partial\Delta^{n_1} * \cdots * \partial\Delta^{n_k}$, and $K_\mathcal{V}$ ($\mathcal{V}$ is the vertex set of $\mathrm{link}_K\sigma$) is q-neighborly with $q \geq 1 + \sum_{i=1}^{k} n_k$, then $H^*(\mathcal{Z}_{\mathrm{S}_\sigma K}) \cong H^*(M)$, where*

$$M = \mathcal{G}(\mathcal{Z}_K) \# \overset{m-n-k}{\underset{j=1}{\sharp}} \binom{m-n-k}{j}(S^{j+2} \times S^{m+n-j-1})$$

$$\# \underset{\substack{I \subset [k], \\ I \neq \varnothing}}{\sharp}\, \overset{m-n-k}{\underset{j=0}{\sharp}} \binom{m-n-k}{j}(S^{\lambda_I+j+2} \times S^{m+n-\lambda_I-j-1}), \quad \lambda_I = \sum_{i \in I}(2n_i + 1).$$

Proof. A straightforward verification shows that $|\sigma| + |\mathcal{V}| = n + k$ and $\dim \mathrm{link}_\sigma K = n - |\sigma| - 1$, so $\dim (\mathrm{link}_\sigma K)_I < q - 1$ for any $I \subset \mathcal{V}$. Since K_I is q-neighborly, then $\pi_k(K) = 0$ for $k < q - 1$. Thus the inclusion map $(\mathrm{link}_K\sigma)_I \to K_I$ is nullhomotopic by Whitehead's theorem. On the other hand, $\mathcal{Z}_{\mathrm{link}_K\sigma} = \prod_{i=1}^{k} S^{2n_i+1}$, so the conditions in Theorem 5.4 are all satisfied. $\qquad\square$

Example 5.7. Let $K = \partial\Delta^{n_1} * \partial\Delta^{n_2}$ with $n_1, n_2 > 2$ (so $\mathcal{Z}_K \approx S^{2n_1+1} \times S^{2n_2+1}$). For any two simplices $\sigma \in \partial\Delta^{n_1}$ and $\tau \in \partial\Delta^{n_2}$, it is easy to see that

$$\mathrm{link}_K(\sigma \cup \tau) = \partial\Delta^{n_1-|\sigma|} * \partial\Delta^{n_2-|\tau|}.$$

So if $\sigma, \tau \neq \varnothing$, and $|\sigma| + |\tau| < n_1 + n_2$, then $\sigma \cup \tau \in K$ satisfies the condition in Theorem 5.4. Note that

$$\mathcal{Z}_{\mathrm{link}_K(\sigma \cup \tau} = \begin{cases} S^{2n_1-2|\sigma|+1} \times S^{2n_2-2|\tau|+1} & \text{for } |\sigma| < n_1,\ |\tau| < n_2, \\ S^{2n_1-2|\sigma|+1} & \text{for } |\tau| = n_2, \\ S^{2n_2-2|\tau|+1} & \text{for } |\sigma| = n_1. \end{cases}$$

Thus by Theorem 5.4, we have the cohomology ring of $\mathcal{Z}_{\mathrm{S}_{\sigma * \tau}K}$ is isomorphic to the one of the connected sum of sphere products

$$S^{2n_1+2} \times S^{2n_2+1} \# S^{2n_1+1} \times S^{2n_2+2} \# S^{2n_1-2|\sigma|+3} \times S^{2n_2+2|\sigma|}$$

$$\# S^{2n_2-2|\tau|+3} \times S^{2n_1+2|\tau|} \# S^{2(n_1+n_2-|\sigma|-|\tau|)+4} \times S^{2(|\sigma|+|\tau|)-1}.$$

Remark 5.8. Actually, it can be proved that the isomorphism in Corollary 5.6 is induced by a homeomorphism of manifolds [32]: $\mathcal{Z}_{\mathrm{S}_{\sigma}K} \approx M$.

The proof of Theorem 5.4 is based on a series of propositions.

Proposition 5.9. *Let K be a simplicial complex with m vertices, $\sigma \in K$, $\mathcal{V}$ be the vertex set of $\mathrm{link}_K \sigma$, $s = |\mathcal{V}| + |\sigma|$. View K as a subcomplex of Δ^{m-1}, and let $L = \Delta^{m-1} \bigcup_{\mathrm{star}_K \sigma} \mathrm{cone}(\mathrm{star}_K \sigma)$. Then the reduced cohomology ring of $\mathcal{Z}_L$ is given by the isomorphism*

$$\widetilde{H}^*(\mathcal{Z}_L; \mathbf{k}) \cong \left(\widetilde{H}^*(\Sigma^2 \mathcal{Z}_{\mathrm{link}_K \sigma}; \mathbf{k}) \otimes \Lambda_{\mathbf{k}}[v_1, \cdots, v_{m-s}]\right) \times \widetilde{H}^*\left(\bigvee_{j=1}^{m-s} \binom{m-s}{j} S^{j+2}; \mathbf{k}\right),$$

where Σ denotes the suspension operation on spaces.

Proof. Let $[m] = \{v_1, \ldots, v_m\}$, $[m+1] = [m] \cup \{v\}$ and $[s] = \{v_{m-s+1}, \ldots, v_m\}$ denote the vertex sets of K, L and $\mathrm{star}_K \sigma$ respectively. It is easy to see that for each $I \in [m+1]$, $I \neq \varnothing$

$$L_I \simeq \begin{cases} \Sigma(\mathrm{link}_K \sigma)_{I \cap [s]} & \text{if } v \in I, \ I \cap \sigma = \varnothing \text{ and } I \cap [s] \neq \varnothing, \\ S^0 & \text{if } \{v\} \subsetneq I \text{ and } I \cap [s] = \varnothing, \\ pt & \text{otherwise.} \end{cases} \quad (5.1)$$

Then we get the additive isomorphism by Theorem 2.12. Clearly the right side of the formula in the proposition has the trivial product structure. From Theorem 2.15 and the fact that $\widetilde{H}^*(L_I) \neq 0 \Rightarrow v \in I$, we have that the left side also has the trivial product structure. Then the proposition follows. $\qquad \square$

Proposition 5.10. *We use the notation as in Proposition 5.9. Suppoe for any $I \subset \mathcal{V}$, the inclusion map $\varphi_I : (\mathrm{link}_K \sigma)_I \to K_I$ is nullhomotopic. Let*

$$K' = K \cup_{\mathrm{star}_K \sigma} \mathrm{cone}(\mathrm{star}_K \sigma).$$

Then the cohomology ring of $\mathcal{Z}_{K'}$ is given by the isomorphism

$$\widetilde{H}^*(\mathcal{Z}_{K'}; \mathbf{k}) \cong \left((\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k}) \otimes \Lambda_{\mathbf{k}}[v]) \times \widetilde{H}^*(\mathcal{Z}_L; \mathbf{k})\right).$$

Proof. Let $[m]$, $[m+1]$ and $[s]$ be as in the proof of 5.9. We assert that for each $I \in [m+1]$, $I \neq \varnothing$

$$K'_I \simeq \begin{cases} K_I \bigvee \Sigma((\text{link}_K\sigma)_{I \cap \mathcal{V}}) & \text{if } v \in I, \ I \cap \sigma = \varnothing \text{ and } I \cap \mathcal{V} \neq \varnothing, \\ K_I \bigvee S^0 & \text{if } \{v\} \subsetneq I \text{ and } I \cap [s] = \varnothing, \\ pt & \text{if } I = \{v\}, \\ K_I & \text{otherwise.} \end{cases} \tag{5.2}$$

The second, third and forth cases are trivial. For the first case, note that K'_I is the mapping cone of the inclusion $\varphi_I : (\text{link}_K\sigma)_I \to K_I$. Therefore, the formula follows from the assumption that φ_I is nullhomotopic.

Let $L' = K_{\mathcal{V}} \cup_{\text{link}_K\sigma} \text{cone}(\text{link}_K\sigma)$. Then for each $I \in [m+1]$

$$L'_I \simeq \begin{cases} \Sigma((\text{link}_K\sigma)_{I \cap \mathcal{V}}) & \text{if } v \in I \text{ and } I \cap \mathcal{V} \neq \varnothing, \\ pt \ \text{ or } \ \varnothing & \text{otherwise.} \end{cases} \tag{5.3}$$

Consider the following commutative diagram of simplicial inclusions

$$\begin{array}{ccc} & L' & \\ & \downarrow{\scriptstyle j_1} \ \searrow{\scriptstyle j_2} & \\ K \xrightarrow{\ i_1\ } & K' \xrightarrow{\ i_2\ } & L, \end{array}$$

which induces a commutative diagram of Hochster algebras

$$\begin{array}{ccc} & \widetilde{\mathcal{H}}^*_{[m+1]}(L') & \\ & \uparrow{\scriptstyle j_1^*} \ \nwarrow{\scriptstyle j_2^*} & \\ \widetilde{\mathcal{H}}^*_{[m+1]}(K) \xleftarrow{\ i_1^*\ } \widetilde{\mathcal{H}}^*(K') \xleftarrow{\ i_2^*\ } & \widetilde{\mathcal{H}}^*(L), \end{array}$$

where i_1, i_2, j_1, j_2 are natural inclusions.

By formula (5.2) and the proof of Proposition 5.9, $\widetilde{\mathcal{H}}^*(K')$ additively splits as $A \oplus B$, where

$$A \cong i_1^*(A) = \operatorname{Im} i_1^*,$$

$$B = \operatorname{Im} i_2^* = \operatorname{Ker} i_1^* \cong \widetilde{\mathcal{H}}^*(L).$$

Now we show that the splitting is multiplicative. B is an ideal of $\widetilde{\mathcal{H}}^*(K')$ is clear. From formulae (5.2) and (5.3), A can be taken as a direct summand of $\operatorname{Ker} j_1^*$, such that $\operatorname{Ker} j_1^*/A \subset \mathcal{H}^1(K')$ and $A = A_1 \oplus A_2$ with $A_1 = \bigoplus_{v \notin I} \widetilde{H}^*(K'_I)$, $A_2 \subset \bigoplus_{v \in I} \widetilde{H}^*(K'_I)$. At first, we prove that A has to be an algebra. Given two elements $a, a' \in A$. Set $a = a_1 + a_2$, $a' = a'_1 + a'_2$, where $a_1, a'_1 \in A_1$ and $a_2, a'_2 \in A_2$. Clearly, $a_1 * a'_1 \in A_1$; $a_2 * a'_2 = 0$. If we can prove that $a_1 * a'_2$, $a_2 * a'_1 \in A$, then $a * a' \in A$, and so A is an algebra. This can be deduced from the fact that $a_1 * a'_2$, $a_2 * a'_1 \in \bigoplus_{i \geq 2} \mathcal{H}^i(K')$ and $\operatorname{Ker} j_1^*/A \subset \mathcal{H}^1(K')$. It remains to show that for any elements $a \in A$ and $b \in B$, $a * b = 0$.

Since B is an ideal, $a * b \in B$. Note that $a * b \in \bigoplus_{i \geq 2} \mathcal{H}^i(K')$, then $a * b = i_2^*(c)$ for some $c \in \bigoplus_{i \geq 2} \mathcal{H}^i(L)$. It is easy to see that j_2^* restricted to $\bigoplus_{i \geq 2} \mathcal{H}^i(L)$ is an isomorphism, so $a * b = 0$ if and only if $j_2^*(c) = j_1^*(a * b) = 0$. Therefore, since $A \subset \operatorname{Ker} j_1^*$, then $j_1^*(a * b) = j_1^*(a) * j_1^*(b) = 0$.

Finally, by the same argument as in the proof of Proposition 4.4, we have

$$A \cong \operatorname{Im} i_1^* \cong G((\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})).$$

The proposition follows immediately. $\qquad\square$

Lemma 5.11. *Suppose there is a short exact sequence of finitely generated abelian groups*

$$0 \to B \to C \to A \to 0.$$

If $C \cong A \oplus B$, then the exact sequence splits.

Proof. If we prove that the exact sequence is zero in $\operatorname{Ext}(A, B)$ (here we view $\operatorname{Ext}(A, B)$ as the set of equivalence classes of extension of A by B), then the lemma follows.

As we know, every finitely generated abelian group can be expressed as

$$G = \mathbb{Z}^m \oplus G_{p_1} \oplus \cdots \oplus G_{p_n},$$

where p_i is prime and G_{p_i} is the p_i-primary component of G, i.e.,

$$G_{p_i} = (\mathbb{Z}_{p_i})^{m_1} \oplus (\mathbb{Z}_{p_i^2})^{m_2} \oplus \cdots \oplus (\mathbb{Z}_{p_i^t})^{m_t}.$$

From homological algebra theory

$$\operatorname{Ext}\left(\bigoplus_i A_i, \bigoplus_j B_j\right) = \bigoplus_i \bigoplus_j \operatorname{Ext}(A_i, B_j)$$

for i and j finite. On the other hand, $\operatorname{Ext}(\mathbb{Z}, G) = 0$ for any group G, and $\operatorname{Ext}(G_p, G_q) = 0$ if $p \neq q$, where p and q are prime. Thus we need only prove the cases that $A = \mathbb{Z}_{p^k}$, and $B = \mathbb{Z}_{p^l}$ or $\mathbb{Z}$.

In the case $A = \mathbb{Z}_{p^k}$ and $B = \mathbb{Z}_{p^l}$, each nonzero class of $\operatorname{Ext}(A, B)$ has the form

$$0 \to \mathbb{Z}_{p^l} \to \mathbb{Z}_{p^{k+l}} \to \mathbb{Z}_{p^k} \to 0.$$

In the case $A = \mathbb{Z}_{p^k}$ and $B = \mathbb{Z}$, each nonzero class of $\operatorname{Ext}(A, B)$ has the form

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{p^k} \to 0.$$

In either case, C is not isomorphic to $A \oplus B$. This completes the proof of the lemma. $\quad\square$

Now let us use the preceding results to complete the proof of Theorem 5.4.

Proof of Theorem 5.4. The cohomology groups with coefficients in $\mathbf{k}$ will be implicit throughout the proof. Let K', $[m]$, $[m+1]$, $[s]$ be the same notations as in Proposition 5.10. Then there are simplicial inclusions $\partial\sigma * \operatorname{link}_K \sigma \xrightarrow{h} \mathrm{S}_\sigma K \xrightarrow{i} K'$ which induce homomorphisms $\widetilde{\mathcal{H}}^*(K') \xrightarrow{i^*} \widetilde{\mathcal{H}}^*(\mathrm{S}_\sigma K) \xrightarrow{h^*} \widetilde{\mathcal{H}}^*_{[m+1]}(\partial\sigma * \operatorname{link}_K \sigma)$.

Consider the homomorphisms of cohomology groups of full subcomplexes

$$i_I^* : \widetilde{H}^j(K_I') \to \widetilde{H}^j((\mathrm{S}_\sigma K)_I), \ I \subseteq [m+1], \ 0 \leq j \leq n-1.$$

We will analyze i_I^* and h_I^* in four cases:

(1) $\sigma \not\subset I$. In this case $K_I' = (S_\sigma K)_I$, so i_I^* is an isomorphism for $0 \le j \le n-1$. Meanwhile, it is easy to verify that $h_I^* = 0$ by the null homotopic assumption, so $\operatorname{Im} i_I^* = \operatorname{Ker} h_I^*$.

(2) $\sigma \subseteq I$ and $I \subsetneq [m]$. In this case $K_I' = K_I = (S_\sigma K)_I \bigcup_{(\partial\sigma*\operatorname{link}_K\sigma)_I} (\operatorname{star}_K\sigma)_I$. Consider the Mayer-Vietoris sequence

$$\cdots \xrightarrow{\delta} \widetilde{H}^j(K_I') \to \widetilde{H}^j((S_\sigma K)_I) \oplus \widetilde{H}^j((\operatorname{star}_K\sigma)_I) \to \widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I)$$

$$\xrightarrow{\delta} \widetilde{H}^{j+1}(K_I') \to \cdots$$

We assert that this long exact sequences breaks up into short exact sequences (note that $\widetilde{H}^*\big((\operatorname{star}_K\sigma)_I\big) = 0$)

$$0 \to \widetilde{H}^j(K_I') \xrightarrow{i_I^*} \widetilde{H}^j((S_\sigma K)_I) \xrightarrow{h_I^*} \widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I) \to 0$$

and these short exact sequences all split. To prove this assertion, first we show that

$$\widetilde{H}^j((S_\sigma K)_I) \cong \widetilde{H}^j(K_I') \oplus \widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I)$$

for all $j \ge 0$. Since $S_\sigma K$ is Gorenstein*, then by applying Alexander duality theorem we have that

$$\widetilde{H}^j((S_\sigma K)_I) \cong \widetilde{H}_{n-j-2}((S_\sigma K)_{[m+1]\setminus I}).$$

Clearly, $(S_\sigma K)_{[m+1]\setminus I} = K_{[m+1]\setminus I}'$. Formula (5.2) implies that

$$\widetilde{H}_{n-j-2}(K_{[m+1]\setminus I}') \cong \widetilde{H}_{n-j-2}(K_{[m]\setminus I}) \oplus \widetilde{H}_{n-j-3}((\operatorname{link}_K\sigma)_{[s]\setminus I}).$$

Since K and $\partial\sigma*\operatorname{link}_K\sigma$ are Gorenstein* complex of dimension $n-1$ and $n-2$ (Proposition 3.5) respectively, and $(\operatorname{link}_K\sigma)_{[s]\setminus I} = (\partial\sigma*\operatorname{link}_K\sigma)_{[s]\setminus I}$, then by applying Alexander duality again, we have that

$$\widetilde{H}_{n-j-2}(K_{[m]\setminus I}) \cong \widetilde{H}^j(K_I) = \widetilde{H}^j(K_I'), \quad \text{and}$$

$$\widetilde{H}_{n-j-3}((\operatorname{link}_K\sigma)_{[s]\setminus I}) \cong \widetilde{H}^j((\partial\sigma*\operatorname{link}_K)_I).$$

Combining these isomorphisms together, we obtain that

$$\widetilde{H}^j((S_\sigma K)_I) \cong \widetilde{H}^j(K_I') \oplus \widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I).$$

Next we prove that i_I^* is injective for all $j \ge 0$. Then the desired result follows by Lemma 5.11. Let G, G' and G'' be the torsion subgroups of $\widetilde{H}^j(K_I')$, $\widetilde{H}^j((S_\sigma K)_I)$ and $\widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I)$ respectively. Then we have $G' \cong G \oplus G''$, and so $|G'| = |G| \cdot |G''|$. Notice that $G \xrightarrow{i'} G' \xrightarrow{h'} G''$ is exact, where i' (resp., h') is the restriction of i_I^* (resp., h_I^*) to G (resp., G'). Thus i' has to be injective. On the other hand, $\operatorname{Ker} i_I^*$ has to be a finite group since

$$\operatorname{rank} \widetilde{H}^j((S_\sigma K)_I) = \operatorname{rank} \widetilde{H}^j(K_I') + \operatorname{rank} \widetilde{H}^j((\partial\sigma*\operatorname{link}_K\sigma)_I).$$

Hence $\operatorname{Ker} i_I^* = \operatorname{Ker} i' = 0$.

(3) $I = [m]$. In this case $\widetilde{H}^*((\mathrm{S}_\sigma K)_I) = 0$, and $\widetilde{H}^*(K_I') = \widetilde{H}^{n-1}(K) \cong \mathbf{k}$. In other words, $i^*(([\mathcal{Z}_K] \otimes 1, 0)) = 0$, where $([\mathcal{Z}_K] \otimes 1, 0)$ is a generator of $\widetilde{H}^{m+n}(\mathcal{Z}_{K'})$ in terms of the isomorphism of Proposition 5.10.

(4) $\{v\} \cup \sigma \in I$. In this case
$$\widetilde{H}^*(K_I', (\mathrm{S}_\sigma K)_I) \cong \widetilde{H}^*\big(\mathrm{cone}\,(\mathrm{star}_K \sigma)_I, \mathrm{cone}\,(\partial\sigma * \mathrm{link}_K \sigma)_I\big) = 0$$

by excision, so i_I^* is an isomorphism. $h_I^* = 0$ is clear.

Combining arguments all above, we have a $\mathbf{k}$-module isomorphism
$$\widetilde{\mathcal{H}}^*(\mathrm{S}_\sigma K) \cong A \oplus B,$$
where $A = \mathrm{Im}\, i^* = \mathrm{Ker}\, h^* \cong \widetilde{H}^*(\mathcal{Z}_{K'})/(([\mathcal{Z}_K] \otimes 1, 0))$ is an ideal of $\widetilde{\mathcal{H}}^*(\mathrm{S}_\sigma K)$,
$$B \subseteq \bigoplus_{\sigma \subseteq I \subsetneq [m]} \widetilde{H}^*((\mathrm{S}_\sigma K)_I);$$
$$B \cong h^*(B) = \mathrm{Im}\, h^* = \bigoplus_{\sigma \subseteq I \subsetneq [m]} \widetilde{H}^*((\partial\sigma * \mathrm{link}_K \sigma)_I).$$

From the proof of Proposition 5.10, we have $A = A_1 \times A_2$, where $A_1 \cong G(\widetilde{H}^*(\mathcal{Z}_K))$, $A_2 \cong \widetilde{H}^*(\mathcal{Z}_L) \subset \bigoplus_{\sigma \cap I = \varnothing} \widetilde{H}^*((\mathrm{S}_\sigma K)_I)$. This implies $A_2 B \subset A_1$, and so A_1 is an ideal of $\widetilde{\mathcal{H}}^*(\mathrm{S}_\sigma K)$. Since $H^*(\mathcal{Z}_{\mathrm{link}_K \sigma})$ is torsion free, both A_2 and B are torsion free.

Now let us verify the ring structure of $\widetilde{H}^*(\mathcal{Z}_{\mathrm{S}_\sigma K})$. We only prove the case that $\mathbf{k} = \mathbb{Z}$, the field case is similar and easier. Since K is Gorenstein*, $\mathcal{Z}_K$ is a manifold. Thus the cup product pairing is nonsingular (see appendix A.1) for $\mathcal{Z}_K$ when torsion in $H^*(\mathcal{Z}_K)$ is factored out. Thus as a direct consequence of Proposition A.4, we can take B such that $A_1 B = 0$. It remains to show the multiplication structure between A_2 and B. If $a_2 \in A_2$, $b \in B$, then $a_2 \in \bigoplus_{v \in I} \widetilde{H}^*((\mathrm{S}_\sigma K)_I)$, and so $a_2 * b \in \bigoplus_{(\{v\} \cup \sigma) \subseteq I} \widetilde{H}^*((\mathrm{S}_\sigma K)_I)$, i.e., $a_2 * b \in A_1$. We assert that if $a_2 \in (A_2)^i$ (here we use the notation $\mathcal{A}^i = \mathcal{A} \cap \mathcal{H}^i$ for a graded subalgebra of $\mathcal{H}^*$), $b \in B^j$ and $a_2 * b \neq 0$, then $i + j = n$. Otherwise, if the order of $a_2 * b$ is infinite, then by nonsingularity of the cup product pairing of A_1, there is a element $a_1 \in (A_1)^{n-i-j}$ such that $a_1 * a_2 * b \neq 0$. Since $a_1 * a_2 = 0$, this is a contradiction. If $a_2 * b$ has finite order m. We consider the homomorphism $\eta : G(\widetilde{H}^*(\mathcal{Z}_K)) \to G(\widetilde{H}^*(\mathcal{Z}_K; \mathbb{Z}_m))$ induced by the map $\mathbb{Z} \to \mathbb{Z}_m$ reducing coefficients mod m. Then the nonsingularity of the cup product pairing for $\mathbb{Z}_m$ coefficient (see Proposition A.3) implies that there is a element $a_1 \in (A_1; \mathbb{Z}_m)^{n-i-j}$ such that $a_1 * \eta(a_2 * b) \neq 0$. However $a_1 * \eta(a_2) = 0$ since Proposition 5.9 holds for any coefficient. Still a contradiction.

It is easy to verify that
$$\mathrm{rank}\,(A_2)^1 = \sum_{j=1}^{m-s} \binom{m-s}{j} = \sum_{j=0}^{m-s-1} \binom{m-s}{j} = \mathrm{rank}\, B^{n-1},$$
and
$$\mathrm{rank}\,(A_2)^i = k_i \cdot \sum_{j=0}^{m-s} \binom{m-s}{j} = k_{n-i} \cdot \sum_{j=0}^{m-s} \binom{m-s}{j} = \mathrm{rank}\, B^{n-i}, \quad \text{for } i > 1,$$

where

$$k_i = \operatorname{rank} \mathcal{H}^{i-1}(\operatorname{link}_K \sigma) = \operatorname{rank} \mathcal{H}^{n-|\sigma|-i+1}(\operatorname{link}_K \sigma) = \operatorname{rank} \mathcal{H}^{n-i}(\partial \sigma * \operatorname{link}_K \sigma)$$

(these equalities follow by formula (5.1) and Alexander duality of $\operatorname{link}_K \sigma$).

Now from the Poincaré duality of $\mathcal{Z}_{S_\sigma K}$, there exists a basis $\{\alpha_{i,r} \mid 1 \le r \le \operatorname{rank}(A_2)^i\}$ of $(A_2)^i$ and a basis $\{\beta_{j,s} \mid 1 \le s \le \operatorname{rank} B^j\}$ of B^j, such that

$$\alpha_{i,r} * \beta_{j,s} = \begin{cases} [\mathcal{Z}_{S_\sigma K}] & \text{if } i+j=n,\, r=s, \\ 0 & \text{otherwise.} \end{cases}$$

The ring isomorphism in Theorem 5.4 follows by a straightforward calculation using all formulae above. □

6. Indecomposability of $\mathcal{Z}_K$ for K a flag 2-sphere

Although the combinatorics of a simplicial complex encodes the topology of the associated moment-angle complex completely, and due to Bahri, Bendersky, Cohen and Gitler [2], the decomposition of $\mathcal{Z}_K$ after a suspension can be entirely described by the geometric realization of the full subcomplexes of K, it is still hard to describe the homeomorphism class or even the homotopy type of $\mathcal{Z}_K$ itself except for some very particular cases of K. For these special cases, the topology of $\mathcal{Z}_K$ has been studied in many works: the simplest case is that $\mathcal{Z}_K$ has the homotopy type of a wedge of spheres [25, 28, 24], and for $\mathcal{Z}_K$ a manifold, the simplest case is that $\mathcal{Z}_K$ has the homeomorphism class of a connected sum of sphere products [34, 23, 18]. Gitler and López de Medrano [23] have shown that there are infinite number of $\mathcal{Z}_K$ which can be decomposed as the connected sum of other nontrivial manifolds (see more examples of moment-angle manifolds with this property in [32]). A natural question is that

Question. For which Gorenstein* complex K, the associated moment-angle manifold $\mathcal{Z}_K$ can not be decomposed as the connected sum of two nontrivial manifolds?

In this section, we answer this question for a special class of simplicial complexes, that is, the flag 2-spheres. We shall begin with some lemmata on the combinatorial and algebraic properties of this kind of simplicial complexes.

Lemma 6.1. *Let K be a triangulation of D^2 with m ($m > 3$) vertices, and let ∂K be the boundary of K, $\mathcal{S}$ be the vertex set of ∂K. If K is flag and $K_\mathcal{S} = \partial K$, then for any $(v_1, v_2) \in MF(K)$, there exists a subset I of $[m]$, such that $\{v_1, v_2\} \subset I$ and K_I is the boundary of a polygon.*

Proof. If $v_1, v_2 \in \mathcal{S}$, then we can choose $I = \mathcal{S}$. Therefore we always assume $\{v_1, v_2\} \not\subset \mathcal{S}$ in the proof.

We prove it by induction on m. Since K is a flag complex, $m = 4$ is impossible, then we star with the case that $m = 5$. In this case K is the join of the boundary of a 4-gon and a vertex, then the statement of the lemma obviously holds. For the induction step,

taking v to be a vertex in $\mathcal{S} \setminus \{v_1, v_2\}$, put $L = \operatorname{star}_K(v)$. Let l_v be the vertex number of L.

First we consider the case $l_v = 4$ (note that $l_v \geq 4$) for some $v \in \mathcal{S} \setminus \{v_1, v_2\}$. In this case L has the form shown in Figure 6.1, where $v', v'' \in \mathcal{S}$. If there is no vertex
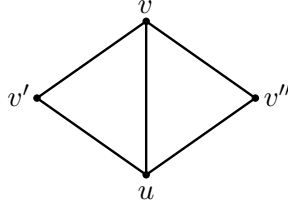

FIGURE 6.1. L with 4 vertices

$u' \in \mathcal{S} \setminus \{v, v', v''\}$ such that $(u, u') \in K$, then it is easy to check that $K' = K_{[m] \setminus \{v\}}$ also satisfies the hypotheses of the lemma. Since K' has $m-1$ vertices, then by induction there exists a subset $I \subset [m] \setminus \{v\}$ such that $(v_1, v_2) \subset I$ and K'_I is the boundary of a polygon. Note that K' is a full subcomplex of K, so $K'_I = K_I$, and so I is the desired subset for K. If there exists a vertex $u' \in \mathcal{S} \setminus \{v, v', v''\}$ such that $(u, u') \in K$, we may assume $|\mathcal{S}| \geq 5$ (otherwise $K = \operatorname{cone} K_{\mathcal{S}}$ is the easy case at the beginning of the proof). Thus there is no vertex v_0 in K such that (v_0, v') and $(v_0, v'') \in K$ (If such v_0 exists, then $v_0 = u'$ and $|\mathcal{S}| = 4$, contradict the assumption that $|\mathcal{S}| \geq 5$). Denote by Δ^2 the 2-dimensional complex consisting of all subsets of $\{v', v'', u\}$, and let

$$K' = (K \setminus L) \cup \Delta^2.$$

Then K' is a triangulation of D^2 with $m-1$ vertices. It is easily verified that K' satisfies all hypotheses of the lemma, hence by induction there exists a subset $I' \subset [m] \setminus \{v\}$ such that $(v_1, v_2) \subset I'$ and $K'_{I'}$ is the boundary of a polygon. If $\{v', v''\} \not\subset I'$, take $I = I'$, otherwise take $I = I' \cup \{v\}$, then I is the desired subset.

Now we consider the case that $l_v \geq 5$ for all $v \in \mathcal{S} \setminus \{v_1, v_2\}$. Firstly, let us give an order on $\mathcal{S}$, e.g. $\mathcal{S} = \{w_1, w_2, \ldots w_n\}$ by counting clockwise, and let $L_i = \operatorname{star}_K(w_i)$, $\mathcal{V}_i$ be the vertex set of L_i, $l_i = |\mathcal{V}_i|$, $1 \leq i \leq n$. So if $w_i \neq v_1, v_2$, then $l_i \geq 5$ and L_i is shown in Figure 6.2.
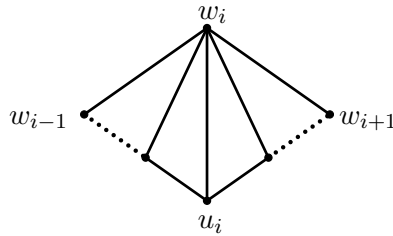

FIGURE 6.2

For $w_i \neq v_1, v_2$, if there are no vertices

$$u_i \in \mathcal{V}_i \setminus \{w_{i-1}, w_i, w_{i+1}\} \quad \text{and} \quad w_{k_i} \in \mathcal{S} \setminus \{w_{i-1}, w_i, w_{i+1}\}$$

such that $(u_i, w_{k_i}) \in K$, then it is easy to see that $K' = K_{[m] \setminus \{w_i\}}$ satisfies the hypotheses of the lemma. Hence by the same arguments as above, we get the desired subset $I \in [m] \setminus \{w_i\}$. Next we prove that this kind of w_i always exists, then we complete the proof of the lemma.

Suppose on the contrary that u_i and w_{k_i} always exist (such that $(u_i, w_{k_i}) \in K$) for each $w_i \neq v_1, v_2$, and suppose without loss of generality that $w_1 \neq v_1, v_2$, then $2 < k_1 < n$. By the assumption $\{v_1, v_2\} \not\subset \mathcal{S}$ at the beginning of the proof, we have

$$\{v_1, v_2\} \cap \{w_1, w_2, \ldots, w_{k_1-1}\} = \varnothing \quad \text{or}$$

$$\{v_1, v_2\} \cap \{w_{k_1+1}, w_{k_1+2}, \ldots, w_n\} = \varnothing.$$

Provided without loss of generality that $\{v_1, v_2\} \cap \{w_1, w_2, \ldots, w_{k_1-1}\} = \varnothing$. Since K_{I_1} (where $I_1 = \{w_1, u_1, w_{k_1}\}$) is a triangulation of D^1, K_{I_1} separates K into two simplicial complexes K_1, K_1' which are both triangulations of D^2. Suppose K_1 is the one contains $\{w_1\}, \{w_2\}, \ldots, \{w_{k_1}\}$. If $u_2 = u_1$, then $k_1 > 3$ by the hypothesis $l_2 \geq 5$ and the flagness of K, thus we can rechoose w_{k_2} if needed such that $w_{k_2} = w_{k_1} > 3$. If $u_2 \neq u_1$, then w_{k_2} must belong to K_1, so $3 < k_2 \leq k_1$. Proceeding inductively, we get that $i+1 < k_i \leq k_{i-1}$ for $1 \leq i \leq k_1 - 1$. But taking $i = k_1 - 1$ we have $k_1 < k_1$, a contradiction. $\qquad\square$

Corollary 6.2. *If K is a flag 2-sphere (with m vertices), then for any $(v_1, v_2) \in MF(K)$, there exists a subset I of $[m]$, such that $\{v_1, v_2\} \subset I$ and K_I is the boundary of a polygon.*

Proof. Choose any vertex $v \in [m] \setminus \{v_1, v_2\}$, then $K_{[m] \setminus \{v\}}$ is apparently a triangulation of D^2 and satisfies the hypotheses in Lemma 6.1. Hence the statement follows immediately.
$\qquad\square$

Lemma 6.3. *If K is a flag 2-sphere, then the Hochster ring $\mathcal{H}^*(K)$ is generated by $\mathcal{H}^1(K)$, i.e., $\widetilde{\mathcal{H}}^*(K)/[\widetilde{\mathcal{H}}^*(K)]^2 = \mathcal{H}^1(K)$.*

Proof. Suppose the vertex set of K is $[m]$. Since $\mathcal{H}^*(K)$ is a Poincaré algebra (note that $\mathcal{H}^*(K)$ is torsion free), then $\mathcal{H}^3(K) = \widetilde{H}^2(K) = \mathbb{Z}$ is generated by $\mathcal{H}^1(K) \oplus \mathcal{H}^2(K)$. Thus we need only prove that $\mathcal{H}^2(K)$ is generated by $\mathcal{H}^1(K)$, i.e., for any $I \subset [m]$, if $a \in \widetilde{H}^1(K_I)$, then $a = \sum_i b_i * c_i$, where $b_i \in \widetilde{H}^0(K_{J_i})$, $c_i \in \widetilde{H}^0(K_{L_i})$ with $J_i \cap L_i = \varnothing$ and $J_i \cup L_i = I$ for each i.

First we prove this for the case $\widetilde{H}^1(K_I) = \mathbb{Z}$ with a generator a'. Let λ be a generator of $\widetilde{H}^2(K) = \mathbb{Z}$, and let $\widehat{I} = [m] \setminus I$. Then by Poincaré duality of $\mathcal{H}^*(K)$, $\widetilde{H}^0(K_{\widehat{I}}) = \mathbb{Z}$ with a generator a so that $a * a' = \lambda$. This implies that $K_{\widehat{I}}$ has two path-components: $K_{\widehat{I}}'$ and $K_{\widehat{I}}''$. Take $\{v_1\} \in K_{\widehat{I}}'$ and $\{v_2\} \in K_{\widehat{I}}''$, then $(v_1, v_2) \in MF(K_{\widehat{I}}) \subset MF(K)$. Hence by Corollary 6.2, there exists a subset $J \subset [m]$, such that $\{v_1, v_2\} \subset J$ and K_J is the boundary of a polygon. Let $K_1 = K_J$, $K_2 = K_{\widehat{I} \cup J}$. Consider the simplicial inclusion

$\varphi : K_1 \to K_2$ which induces a homomorphism of Hochster rings

$$\varphi_{\mathcal{H}}^* : \mathcal{H}^*(K_2) \to \mathcal{H}_{\widehat{I} \cup J}^*(K_1)$$

(View K_1 as a simplicial complex on $\widehat{I} \cup J$). It is clear that

$$\varphi_{\widehat{I}}^* : \widetilde{H}^0((K_2)_{\widehat{I}}) \to \widetilde{H}^0((K_1)_{\widehat{I}})$$

is a monomorphism (in fact $\mathrm{Im}\varphi_{\widehat{I}}^*$ is a direct summand of $H^0((K_1)_{\widehat{I}})$) and

$$\varphi_{J \setminus \widehat{I}}^* : \widetilde{H}^0((K_2)_{J \setminus \widehat{I}}) \to \widetilde{H}^0((K_1)_{J \setminus \widehat{I}})$$

is an isomorphism. Since $\mathcal{H}_{\widehat{I} \cup J}^*(K_1)$ is a Poincaré algebra (note K_1 is a triangulation of S^1), there exists an element b of $\widetilde{H}^0((K_1)_{J \setminus \widehat{I}})$ such that $\varphi_{\widehat{I}}^*(a) * b = \xi$, where ξ is a generator of $\widetilde{H}^1(K_1) = \mathbb{Z}$. This implies that $a * (\varphi_{J \setminus \widehat{I}}^*)^{-1}(b)$ is one of the generators of $\widetilde{H}^1(K_2)$ ($\varphi_{\mathcal{H}}^*$ is surjective on $\mathcal{H}^2$). Thus by applying the Poincaré duality of $\mathcal{H}^*(K)$, there is an element $c \in \widetilde{H}^0(K_{I'})$, where $I' = [m] \setminus (\widehat{I} \cup J)$, such that

$$a * (\varphi_{J \setminus \widehat{I}}^*)^{-1}(b) * c = \lambda = a * a'.$$

It follows that

$$(\varphi_{J \setminus \widehat{I}}^*)^{-1}(b) * c = a'.$$

Then we get the desired result.

For the general case $\widetilde{H}^1(K_I) = \mathbb{Z}^n$, similarly we have $\widetilde{H}^0(K_{\widehat{I}}) = \mathbb{Z}^n$. So $K_{\widehat{I}}$ has $n + 1$ path-components, say $L_0, L_1, \ldots, L_n$. Give a basis $\{a_i\}_{1 \le i \le n}$ of $\widetilde{H}^0(K_{\widehat{I}})$ defined by $a_i = \sum_{(v) \in L_i} (v)$. It determines a dual basis $\{a_i'\}_{1 \le i \le n}$ of $\widetilde{H}^1(K_I)$, i.e., $a_i * a_j' = \lambda$ for $i = j$ and zero otherwise. If $(v_0, v_1) \in MF(K_{\widehat{I}}) \subset MF(K)$ with $\{v_0\} \in L_0$ and $\{v_1\} \in L_1$, then by applying Corollary 6.2 agian, there exists a subset $J \subset [m]$ such that $(v_0, v_1) \in J$ and K_J is the boundary of a polygon. As in the preceding paragraph, let $K_1 = K_J$, $K_2 = K_{\widehat{I} \cup J}$. Consider the simplicial inclusion $\varphi : K_1 \to K_2$ which induces a homomorphism of Hochster rings

$$\varphi_{\mathcal{H}}^* : \mathcal{H}^*(K_2) \to \mathcal{H}_{\widehat{I} \cup J}^*(K_1).$$

In similar fation there exists an element b of $\widetilde{H}^0((K_1)_{J \setminus \widehat{I}})$ such that $\varphi_{\widehat{I}}^*(a_i) * b = \xi$, where ξ is a generator of $\widetilde{H}^1(K_1) = \mathbb{Z}$, for $i = 1$ and zero otherwise. This implies that $a_1 * (\varphi_{J \setminus \widehat{I}}^*)^{-1}(b)$ is one of the generators of $\widetilde{H}^1(K_2)$. Appealing to the Poincaré duality of $\mathcal{H}^*(K)$, there is an element $c \in \widetilde{H}^0(K_{I'})$ (where $I' = [m] \setminus (\widehat{I} \cup J)$) such that

$$a_1 * (\varphi_{J \setminus \widehat{I}}^*)^{-1}(b) * c = \lambda = a_1 * a_1' \quad \text{and}$$

$$a_i * (\varphi_{J \setminus \widehat{I}}^*)^{-1}(b) * c = 0 \quad \text{for } i \ne 1.$$

It follows that $(\varphi_{J \setminus \widehat{I}}^*)^{-1}(b) * c = a_1'$, and so a_1' is generated by $\mathcal{H}^1(K)$. Similarly, a_i' is generated by $\mathcal{H}^1(K)$ for each i, and the lemma follows. $\qquad \square$

Note that Lemma 6.3 dose not hold for general flag complexes. To see this, consider the following example:

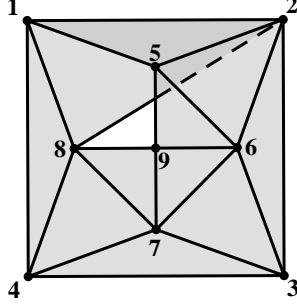**Example 6.4.** Let K be a 2-dimensional flag complex shown in Figure 6.3.

FIGURE 6.3

K contains nine vertices and eleven 2-simplices:

$$\{(1,2,5),(1,2,8),(1,4,8),(2,3,6),(2,5,6),(3,4,7),$$
$$(3,6,7),(4,7,8),(5,6,9),(6,7,9),(7,8,9)\}.$$

It is easy to see that $|K| \simeq S^1$, so $\widetilde{H}^1(K) \cong \mathbb{Z}$. However $\widetilde{H}^1(K) = \mathcal{H}^2(K)$ is not generated by $\mathcal{H}^1(K) = \widetilde{H}^0(K)$, since a straightforward observation shows that for any division $I \cup J = [9]$, $\widetilde{H}^0(K_I) = 0$ or $\widetilde{H}^0(K_J) = 0$.

Definition 6.5. A ring $\mathcal{R}$ is called *decomposable* if there exists nonzero rings $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R} \cong \mathcal{R}_1 \times \mathcal{R}_2$. Otherwise, $\mathcal{R}$ is called *indecomposable*. If $\mathcal{R}$ is a graded ring, and the corresponding graded decomposition exists, then $\mathcal{R}$ is called *graded decomposable*. Otherwise, $\mathcal{R}$ is called *graded indecomposable*.

A **k**-algebra A is called *decomposable* if there exists nonzero **k**-algebra A_1 and A_2 such that $A \cong A_1 \times A_2$. Otherwise, A is called *indecomposable*. The others are defined similarly for **k**-algebras.

Theorem 6.6. *Let K be a flag 2-sphere. Then $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is a graded indecomposable ring, where $[\mathcal{Z}_K]$ is the top class of $\widetilde{H}^*(\mathcal{Z}_K)$.*

Definition 6.7. An n-dimensional manifold M^n is called *prime* if $M^n = M_1^n \# M_2^n$ implies $M_1 = S^n$ or $M_2 = S^n$.

The following theorem as a consequence of Theorem 6.6 answers the question at the beginning of this section for flag 2-spheres.

Theorem 6.8. *Let K be a flag 2-sphere. Then $\mathcal{Z}_K$ is a prime manifold.*

Proof. Suppose $\mathcal{Z}_K = M_1 \# M_2$. Since $\mathcal{Z}_K$ is always simply connected (cf. [11, Corollary 6.19]), then by Van Kampen's theorem M_1 and M_2 are both simply connected. Hence by

Poincaré conjecture, if M_i has the homology of a sphere, then it actually homeomorphic to the sphere. Note that

$$\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K]) = \widetilde{H}^*(M_1)/([M_1]) \times \widetilde{H}^*(M_2)/([M_2]).$$

Then the theorem follows from Theorem 6.6 immediately. $\qquad\square$

The proof of Theorem 6.6 is separated into several lemmata, in which we always assume that K is a flag 2-sphere with vertex set $[m]$.

Lemma 6.9. *If there is an (ungraded) isomorphism of rings*

$$\phi : \widetilde{\mathcal{H}}^*(K)/\mathcal{H}^3(K) \xrightarrow{\cong} \mathcal{R}_1 \times \mathcal{R}_2$$

for two nonzero rings $\mathcal{R}_1$ and $\mathcal{R}_2$, then there exist two subset I_1, $I_2 \in [m]$ such that K_{I_1} and K_{I_2} are triangulations of S^1, and $\phi(\widetilde{H}^1(K_{I_i})) \subset \mathcal{R}_i$, $i = 1, 2$.

Proof. Note first that

$$\widetilde{\mathcal{H}}^*(K)/\mathcal{H}^3(K) = \mathcal{H}^1(K) \oplus \mathcal{H}^2(K) \text{ (as group)}.$$

Let $p : \mathcal{H}^1(K) \oplus \mathcal{H}^2(K) \to \mathcal{H}^1(K)$ be the projection map. We claim that p restricted to $\mathcal{R}_1$ and $\mathcal{R}_2$ are both nonzero. Otherwise suppose $p \mid \mathcal{R}_1 = 0$, then $p \mid \mathcal{R}_2$ must be surjective. Since by Lemma 6.3, $\widetilde{\mathcal{H}}^*(K)/\mathcal{H}^3(K)$ is generated by $\mathcal{H}^1(K)$, then $\widetilde{\mathcal{H}}^*(K)/\mathcal{H}^3(K)$ is generated by $\mathcal{R}_2$, a contradiction.

Suppose $r_1 \in \mathcal{R}_1$ such that $p_I\phi^{-1}(r_1) \neq 0$ for some $I \subset [m]$, where p_I is the composition of p and the projection

$$p'_I : \mathcal{H}^1(K) = \bigoplus_{I \in [m]} \widetilde{H}^0(K_I) \to \widetilde{H}^0(K_I).$$

Assume $(v_1, v_2) \in MF(K_I) \subset MF(K)$. Thus by Lemma 6.1, there exists a subset $J \subset [m]$ such that $(v_1, v_2) \subset J$ and K_J is the boundary of a polygon. Applying the same argument as in the proof of Lemma 6.3, there exists an element $a \in \widetilde{H}^0(K_{J\setminus I})$ such that $a * p_I\phi^{-1}(r_1) \neq 0$. Hence $a * \phi^{-1}(r_1) \neq 0$. If $\phi(a) = \gamma_1 + \gamma_2$ with $\gamma_1 \in \mathcal{R}_1$ and $\gamma_2 \in \mathcal{R}_2$, then we can write

$$\phi^{-1}(\gamma_1) = a_1 + b, \quad \phi^{-1}(\gamma_2) = a_2 - b$$

with $a_1, a_2 \in \widetilde{H}^0(K_{J\setminus I})$, $a_1 + a_2 = a$ and $b \in \bigoplus_{I' \neq J\setminus I} \widetilde{H}^*(K_{I'})$. Set $b = b_1 + b_2$, where $b_1 \in \mathcal{H}^1(K)$, $b_2 \in \mathcal{H}^2(K)$.

If $b_1 = 0$, then we claim that $a_1 = a$ and $a_2 = 0$. Otherwise $a_1 = 0$ and $a_2 = a$ or $a_1, a_2 \neq 0$. In the first case, $a * \phi^{-1}(r_1) = \phi^{-1}(\gamma_2) * \phi^{-1}(r_1) \neq 0$ (note $\mathcal{H}^1(K) * \mathcal{H}^2(K) = 0$ in $\widetilde{\mathcal{H}}^*(K)/\mathcal{H}^3(K)$), so we have $\gamma_2 r_1 \neq 0$, contrary to the fact that $\mathcal{R}_1\mathcal{R}_2 = 0$. For the second case, from the Poincaré duality of $\mathcal{H}^*(K_J)$, there exist two elements $a'_1, a'_2 \in \widetilde{H}^0(K_{I\cap J})$ such that $a_1 * a'_1 = a_2 * a'_2 \neq 0$ in $\widetilde{H}^1(K_J) = \mathbb{Z}$, so

$$\phi^{-1}(\gamma_1) * a'_1 = \phi^{-1}(\gamma_2) * a'_2 \neq 0,$$

and so $\gamma_1\phi(a_1') = \gamma_2\phi(a_2') \neq 0$. This implies that $\mathcal{R}_1 \cap \mathcal{R}_2 \neq 0$, a contradiction. When $b \in \mathcal{H}^2(K)$ and $a = a_1$, then from the fact that $a * a' \neq 0$ in $\widetilde{H}^1(K_J) = \mathbb{Z}$ for some $a' \in \widetilde{H}^0(K_{I\cap J})$ and

$$\phi(a * a') = \phi((a+b) * a') = \gamma_1\phi(a') \in \mathcal{R}_1,$$

we get the desired result $\phi(\widetilde{H}^1(K_J)) \subset \mathcal{R}_1$ (put $I_1 = J$). Next we prove that b_1 must be zero.

If $b_1 \neq 0$, without loss of generality we assume $b = b_1 \in \widetilde{H}^0(K_{I'})$ for some $I' \neq J \setminus I$, then there exists an element $b' \in \widetilde{H}^1(K_{\widehat{I'}})$, where $\widehat{I'} = [m] \setminus I'$, such that $b * b' \neq 0$ in $\widetilde{H}^2(K) = \mathbb{Z}$ by the Poincaré duality of $\mathcal{H}^*(K)$. Lemma 6.3 implies that $b' = \sum_i c_i * e_i$ with $c_i \in \widetilde{H}^0(K_{J_i})$, $e_i \in \widetilde{H}^0(K_{J_i'})$, $J_i \cap J_i' = \varnothing$ and $J_i \cup J_i' = \widehat{I'}$ for each i. Thus $b * c_i * e_i \neq 0$ for some i. Since $I' \neq J \setminus I$, then $\widehat{I'} \cap (J \setminus I) \neq \varnothing$, and then $J_i \cap (J \setminus I) \neq \varnothing$ or $J_i' \cap (J \setminus I) \neq \varnothing$. Assume $J_i \cap (J \setminus I) \neq \varnothing$, then

$$\phi^{-1}(\gamma_1) * c_i = (a_1 + b) * c_i = b * c_i = (b - a_2) * c_i = -\phi^{-1}(\gamma_2) * c_i \neq 0.$$

This implies that $\phi^{-1}(\mathcal{R}_1) \cap \phi^{-1}(\mathcal{R}_2) \neq 0$, a contradiction. So $b_1 = 0$.

Similarly, by considering $r_2 \in \mathcal{R}_2$ such that $p_I\phi^{-1}(r_2) \neq 0$ for some $I \subset [m]$, we get a subset $I_2 \subset [m]$ such that $\phi(\widetilde{H}^1(K_{I_2})) \subset \mathcal{R}_2$ and K_{I_2} a triangulation of S^1. $\square$

Lemma 6.10. *If there is a graded isomorphism*

$$\phi : \widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K]) \xrightarrow{\cong} \mathcal{R}_1 \times \mathcal{R}_2$$

for two nonzero graded rings $\mathcal{R}_1$ and $\mathcal{R}_2$, and if there is a subset $I \subset [m]$ such that K_I is the boundary of a polygon and $\phi(\widetilde{H}^1(K_I)) \subset \mathcal{R}_1$ (here $\widetilde{H}^1(K_I)$ is viewed as a subgroup of $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ under the isomorphism given in Theorem 2.12), then for any $\sigma \in MF(K_I)$, $\phi(\widetilde{H}^0(K_\sigma)) \subset \mathcal{R}_1$.

Proof. Let λ_σ be a generator of $\widetilde{H}^0(K_\sigma) = \mathbb{Z}$ for each $\sigma \in MF(K)$. Set $\phi(\lambda_\sigma) = r_\sigma' + r_\sigma''$ with $r_\sigma' \in \mathcal{R}_1$ and $r_\sigma'' \in \mathcal{R}_2$. Since ϕ preserves grading and each element of $\widetilde{H}^3(\mathcal{Z}_K)$ belongs to $\bigoplus_{\sigma \in MF(K)} \widetilde{H}^0(K_\sigma)$, then we can suppose

$$\phi^{-1}(r_\sigma') = \sum_{\tau \in MF(K)} k_\tau \cdot \lambda_\tau, \quad \phi^{-1}(r_\sigma'') = \sum_{\tau \in MF(K)} l_\tau \cdot \lambda_\tau, \quad k_\tau, l_\tau \in \mathbb{Z}.$$

Since $\phi^{-1}(r_\sigma') + \phi^{-1}(r_\sigma'') = \lambda_\sigma$, then

$$k_\sigma + l_\sigma = 1, \quad \text{and} \quad k_\tau = -l_\tau \text{ for } \tau \neq \sigma.$$

If $\sigma \in MF(K_I)$, first we prove that $l_\sigma = 0$. Suppose on the contrary that $l_\sigma \neq 0$, then $\phi^{-1}(r_\sigma'') * \lambda' \neq 0$ for some $\lambda' \in \widetilde{H}^0(K_{I\setminus\sigma}) = \mathbb{Z}$. Set $\phi(\lambda') = r_1 + r_2$ with $r_1 \in \mathcal{R}_1$, $r_2 \in \mathcal{R}_2$. Since $\phi^{-1}(r_\sigma'') * \lambda' \neq 0$, we have $r_\sigma''\phi(\lambda') = r_\sigma''(r_1 + r_2) \neq 0$, and then $r_\sigma'' r_2 \neq 0$. By the same argument as in the proof of Lemma 6.9, we have $\phi^{-1}(r_2) = \lambda' + b$ for some $b \in \mathcal{H}^2(K)$. Now, since $\lambda_\sigma * (\lambda' + b) \neq 0 \in \widetilde{H}^1(K_I)$ ($\lambda_\sigma * b = 0$ in $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ for dimension reasons), then

$$\phi(\lambda_\sigma)\phi(\lambda' + b) \neq 0 \in \phi(\widetilde{H}^1(K_I)) \subset \mathcal{R}_1.$$

On the other hand, notice that $\mathcal{R}_2$ is an ideal of $\mathcal{R}_1 \times \mathcal{R}_2$, so

$$\phi(\lambda_\sigma)\phi(\lambda' + b) = \phi(\lambda_\sigma)r_2 \in \mathcal{R}_2.$$

This implies that $\mathcal{R}_1 \cap \mathcal{R}_2 \neq 0$, a contradiction.

Next we prove that $k_\tau = l_\tau = 0$ for all $\tau \neq \sigma$. Let $\lambda_{\widehat{\tau}}$ be a generator of $\widetilde{H}^1(K_{\widehat{\tau}}) = \mathbb{Z}$, where $\widehat{\tau} = [m] \setminus \tau$. Then $\lambda_\tau * \lambda_{\widehat{\tau}}$ is a generator of $\widetilde{H}^2(K)$. By Lemma 6.3, $\lambda_{\widehat{\tau}} = \sum_i c_i * e_i$ with $c_i \in \widetilde{H}^0(K_{J_i})$, $e_i \in \widetilde{H}^0(K_{J_i'})$, $J_i \cap J_i' = \varnothing$ and $J_i \cup J_i' = \widehat{\tau}$ for each i. Thus $\lambda_\tau * c_i * e_i \neq 0$ (in $\widetilde{\mathcal{H}}^*(K)$) for some i. Since $\tau \neq \sigma$, then $\widehat{\tau} \cap \sigma \neq \varnothing$, and then $J_i \cap \sigma \neq \varnothing$ or $J_i' \cap \sigma \neq \varnothing$. Assume $J_i \cap \sigma \neq \varnothing$, then $\lambda_\sigma * c_i = 0$. So if $l_\tau \neq 0$ for some $\tau \neq \sigma$, then

$$\phi^{-1}(r_\sigma') * c_i = -\phi^{-1}(r_\sigma'') * c_i \neq 0.$$

This implies that $\phi^{-1}(\mathcal{R}_1) \cap \phi^{-1}(\mathcal{R}_2) \neq 0$, a contradiction. So $\phi(\lambda_\sigma) = r_\sigma' \in \mathcal{R}_1$, and the lemma follows. $\qquad\square$

Lemma 6.11. *If $\sigma_1, \sigma_2 \in MF(K)$ and $\sigma_1 \cap \sigma_2 = \{v\}$, then there exist three subsets $I_1, I_2, I_3 \subset [m]$ such that K_{I_j} is the boundary of a polygon, $j = 1, 2, 3$, $\sigma_i \in MF(K_{I_i})$ and $MF(K_{I_3}) \cap MF(K_{I_i}) \neq \varnothing$, $i = 1, 2$.*

Proof. Suppose that I_1 and I_2 have been taken such that $\sigma_i \in MF(I_i)$ and K_{I_i} is a triangulation of S^1, $i = 1, 2$ (Lemma 6.1 guarantees the existence of I_1 and I_2). If $MF(I_1) \cap MF(I_2) \neq \varnothing$, we can take $I_3 = I_1$. Otherwise, let I_3 be the vertex set of $\mathrm{link}_K(v)$, then K_{I_3} is a triangulation of S^1 by the flagness of K. It is easily verified that $MF(K_{I_3}) \cap MF(K_{I_i}) \neq \varnothing$, $i = 1, 2$. $\qquad\square$

Definition 6.12. A simplicial complex K is called a *suspension complex*, if $K = \Sigma L$ $(\Sigma L = S^0 * L)$ for some complex L.

Lemma 6.13. *If K is not a suspension complex, then for any two different missing faces $\sigma_1, \sigma_2 \in MF(K)$, there is a sequence of missing faces $\tau_1, \tau_2, \ldots, \tau_n$ such that $\tau_1 = \sigma_1$, $\tau_n = \sigma_2$ and $\tau_i \cap \tau_{i+1} \neq \varnothing$, $i = 1, \ldots, n-1$.*

Proof. If $\sigma_1 \cap \sigma_2 \neq \varnothing$, there is nothing to prove, so we assume $\sigma_1 \cap \sigma_2 = \varnothing$. Set $\sigma_1 = (v_1, v_1')$, $\sigma_2 = (v_2, v_2')$. If $\sigma_2 \notin MF(\mathrm{link}_K(v_1)) \cap MF(\mathrm{link}_K(v_1'))$, then one of (v_1, v_2), (v_1, v_2'), (v_1', v_2), (v_1', v_2') is a missing face of K, say $(v_1, v_2) \in MF(K)$. We then get the desired sequence: $\tau_1 = \sigma_1, \tau_2 = (v_1, v_2), \tau_3 = \sigma_2$. Similarly we can prove the case that $\sigma_1 \notin MF(\mathrm{link}_K(v_2)) \cap MF(\mathrm{link}_K(v_2'))$.

If $\sigma_2 \in MF(\mathrm{link}_K(v_1)) \cap MF(\mathrm{link}_K(v_1'))$, since K is not a suspension complex, there exists a vertex u of $\mathrm{link}_K(v_1)$ such that (v_1', u) is a missing face of K. Take a missing face $\sigma_3 \in MF(\mathrm{link}_K(v_1))$ so that $u \in \sigma_3$. Since $\mathrm{link}_K(v_1)$ is a triangulation of S^1, thus if $\mathrm{link}_K(v_1)$ has more than 5 vertices, then σ_2 and σ_3 can obviously be connected by a sequence of missing faces of $\mathrm{link}_K(v_1)$, which satisfies the condition in the lemma. Thus we get the desired sequence again:

$$\tau_1 = \sigma_1, \tau_2 = (v_1', u), \tau_3 = \sigma_3, \ldots, \tau_n = \sigma_2.$$

The remaining case is that:

$$\sigma_2 \in MF(\mathrm{link}_K(v_1)) \cap MF(\mathrm{link}_K(v_1')) \quad \text{and}$$

$$\sigma_1 \in MF(\mathrm{link}_K(v_2)) \cap MF(\mathrm{link}_K(v_2')) \quad \text{and}$$

$$\mathrm{link}_K(v_1) \cong \mathrm{link}_K(v_1') \cong \mathrm{link}_K(v_2) \cong \mathrm{link}_K(v_2') \cong S^0 * S^0.$$

It is easy to see that $K \cong S^0 * S^0 * S^0$ in this case, contrary to the hypotheses, so the lemma holds. $\qquad\square$

Now let us use the preceding results to complete the proof of Theorem 6.6.

proof of Theorem 6.6. Suppose on the contrary that there is a graded ring isomorphism

$$\phi : \widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K]) \xrightarrow{\cong} \mathcal{R}_1 \times \mathcal{R}_2$$

for two nonzero graded rings $\mathcal{R}_1$ and $\mathcal{R}_2$. Then by Lemma 6.9, there exist two subset $I_1, I_2 \in [m]$ such that K_{I_1} and K_{I_2} are triangulations of S^1, and $\phi(\widetilde{H}^1(K_{I_1})) \subset \mathcal{R}_1$, $\phi(\widetilde{H}^1(K_{I_2})) \subset \mathcal{R}_2$. Lemma 6.10 implies that $\phi(\widetilde{H}^0(K_{\sigma_1})) \subset \mathcal{R}_1$ (resp. $\phi(\widetilde{H}^0(K_{\sigma_2})) \subset \mathcal{R}_2$.) for each $\sigma_1 \in MF(K_{I_1})$ (resp. $\sigma_2 \in MF(K_{I_2})$).

If K is not a suspension complex, Lemma 6.13 says that $\sigma_1 \in MF(K_{I_1})$ and $\sigma_2 \in MF(K_{I_2})$ can be connected by a sequence of missing faces $\tau_1, \tau_2, \ldots, \tau_n$ such that $\tau_1 = \sigma_1$, $\tau_n = \sigma_2$ and $\tau_i \cap \tau_{i+1} \neq \varnothing$, $i = 1, \ldots, n-1$. Let λ_{τ_i} be a generator of $\widetilde{H}^0(K_{\tau_i}) = \mathbb{Z}$. Then by combining the conclusion of Lemma 6.10 and Lemma 6.11 together, we have that

$$\phi(\lambda_{\tau_i}) \in \mathcal{R}_1 \cap \mathcal{R}_2 \quad \text{for all } 1 \leq i \leq n.$$

This is a contradiction.

If K is a suspension complex, i.e., $K = S^0 * L$ so that L is the boundary of a m-gon $(m > 3)$, then

$$\mathcal{Z}_K \approx S^3 \times (\overset{m-2}{\underset{j=1}{\#}} j\binom{m-1}{j+1} S^{j+2} \times S^{m-j}).$$

Hence $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is clearly indecomposable. $\qquad\square$

Following the same procedure as in the proof of Theorem 6.6, we get the following corollary which is needed later.

Corollary 6.14. *If K is a flag 2-sphere, then $\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})/([\mathcal{Z}_K])$ is a graded indecomposable $\mathbf{k}$-algebra.*

When K is a flag n-sphere with $n \geq 3$, it seems that $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is also an indecomposable ring (we can not find a counterexample), although the arguments in the proof for the case $n = 2$ can not be applied to this case exactly. So we give the following

Conjecture 6.15. *If K is a flag n-dimensional $(n \geq 3)$ Gorenstein* complex, then $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is an indecomposable ring, and consequently $\mathcal{Z}_K$ is a prime manifold.*

Note that if a simplicial 2-sphere K is not flag and $K \neq \partial\Delta^3$ (which means that K is the connected sum of two simplicial 2-spheres), then $\widetilde{H}^*(\mathcal{Z}_K)/([\mathcal{Z}_K])$ is not an indecomposable ring by Corollary 4.8. Hence when K is a nontrivial (i.e., $K \neq \partial\Delta^3$) simplicial 2-sphere, the flagness of K is an 'if and only if' condition for $\mathcal{Z}_K$ to be prime. Naturally, We may propose the following

Problem 6.16. Find a 'if and only if' condition such that $\mathcal{Z}_K$ is prime for higher dimensional Gorenstein* K.

7. Decomposition of $H^*(\mathcal{Z}_K)$ for simplicial 2-spheres

In this section, we give a brief discussion about the cohomological rigidity of moment-angle manifolds associated to simplicial 2-spheres (see [19] for a more specific study about this topic). Cohomology ring does not distinguish closed smooth manifolds in general but for moment-angle manifolds, based on the special toric action on $\mathcal{Z}_K$ and by checking some examples, some toric topologist proposed the following (cf. [9, 15]):

Question A. Suppose $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ are two moment-angle manifolds such that

$$H^*(\mathcal{Z}_{K_1}) \cong H^*(\mathcal{Z}_{K_2})$$

as graded rings. Are $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_1}$ homeomorphic?

The moment-angle manifolds giving the positive answer to the question are called *cohomologically rigid*. Buchstaber asked another question in his lecture note [9, Problem 7.6] (here we restrict his question to Gorenstein* complexes):

Question B. Let K_1 and K_2 be two Gorenstein* complexes (may have different dimension), and let $\mathcal{Z}_{K_1}$ and $\mathcal{Z}_{K_2}$ be their respective moment-angle manifolds. When a graded ring isomorphism $H^*(\mathcal{Z}_{K_1}; \mathbf{k}) \cong H^*(\mathcal{Z}_{K_2}; \mathbf{k})$ implies a combinatorial equivalence $K_1 \approx K_2$?

Let us call the Gorenstein* complexes giving the positive answer to the question B-*rigid over* $\mathbf{k}$ (if $\mathbf{k} = \mathbb{Z}$, simply refer to it as B-*rigid*). It is clear that K is B-rigid implies that $\mathcal{Z}_K$ is cohomologically rigid.

Definition 7.1. If a simplicial $(n-1)$-sphere K can be expressed as a connected sum of two simplicial spheres, then K is called *reducible*. Otherwise, K is called *irreducible*. Furthermore if K is irreducible and $K \neq \Delta^n$, then K is called *essentially irreducible*.

Hence any simplicial sphere K can be decomposed into

$$K = K_1 \# K_2 \#, \dots, \# K_n$$

such that each K_i is irreducible, and it is easily verified that this decomposition is unique up to a permutation of $[n]$. In particular if K is a simplicial 2-sphere, then the condition that K is flag is equivalent to saying K to be *essentially irreducible*. From Theorem 4.2 we get the following

Proposition 7.2. *Let* $K \in \mathcal{C}(K_1 \# K_2 \# \cdots \# K_m)$ *for irreducible simplicial* n-*spheres* $K_1, K_2, \ldots, K_m$. *If* K *is* B-*rigid, then* K *is the only element in*

$$\mathcal{C}(K_1 \# K_2 \# \cdots \# K_m).$$

For the case K is a simplicial 2-sphere, Choi and Kim [14] gave a "$\Longleftrightarrow$" condition for K to be the only element in $\mathcal{C}(K_1 \# K_2 \# \cdots \# K_m)$ by using a result of Fleischner and Imrich [20, Theorem 3]. The following notation is adopted from [14]. Let T_4, C_8, O_6, D_{20} and I_{12} be the boundary of the five Platonic solids (subscripts indicating vertex numbers): the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron respectively; $\xi_1(C_8)$ and $\xi_1(D_{20})$ be the *first-subdivision* of C_8 and D_{20} resp. (see Figure 7.1); $\xi_2(C_8)$ and $\xi_2(D_{20})$ be the *second-subdivision* of C_8 and D_{20} resp. (see Figure 7.2); B_n be the suspension of the boundary of a $(n-2)$-gon, called a *bipyramid* (see Figure 7.3).
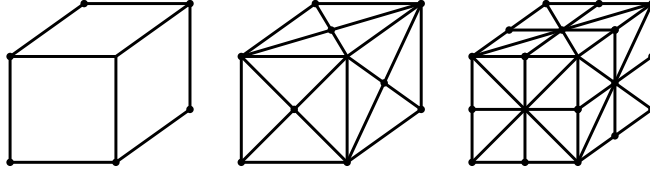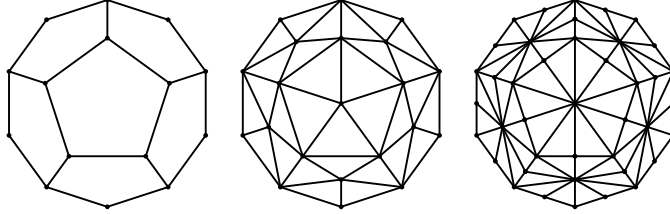

FIGURE 7.1. C_8, $\xi_1(C_8)$ and $\xi_2(C_8)$.

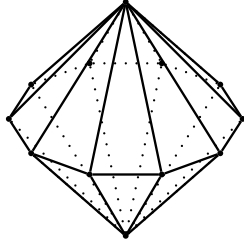FIGURE 7.2. D_{20}, $\xi_1(D_{20})$ and $\xi_2(D_{20})$.

FIGURE 7.3. B_n

Theorem 7.3 ([14, Theorem 1.2]). *Let K be a reducible simplicial 2-sphere. If K is the only element in $\mathcal{C}(K_1 \# K_2 \# \cdots \# K_m)$ (where each K_i is irreducible), then $K = T_4 \# T_4 \# T_4$ or $K = K_1 \# K_2$, where*

$$K_1 \in \{T_4, O_6, I_{12}\},$$
$$K_2 \in \{T_4, O_6, I_{12}, \xi_1(C_8), \xi_2(C_8), \xi_1(D_{20}), \xi_2(D_{20})\} \cup \{B_n : n \geq 7\}.$$

In the remainder, we focus our attention on the simplicial 2-spheres, starting with the following theorem.

Theorem 7.4. *Let K and K' be simplicial 2-spheres. If*

$$K = K_1 \# K_2 \#, \ldots, \# K_n, \quad K' = K'_1 \# K'_2 \#, \ldots, \# K'_{n'}$$

such that each K_i and K'_i are irreducible, and if $H^*(\mathcal{Z}_K; \mathbf{k}) \cong H^*(\mathcal{Z}_{K'}; \mathbf{k})$ (as graded rings), then $n = n'$ and there is a permutation $j \curvearrowright j'$ such that

$$H^*(\mathcal{Z}_{K_j}; \mathbf{k})/([\mathcal{Z}_{K_j}]) \cong H^*(\mathcal{Z}_{K'_{j'}}; \mathbf{k})/([\mathcal{Z}_{K'_{j'}}]), \ 1 \leq j \leq n.$$

We first separate off a

Lemma 7.5. *Let K be a flag 2-sphere, then $G^n(\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k}))/([\mathcal{Z}_K] \otimes v_{[n]})$, $n \geq 1$, is a graded indecomposable $\mathbf{k}$-algebra.*

Proof. Note that

$$G^n(\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k}))/([\mathcal{Z}_K] \otimes v_{[n]}) \cong (\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})/([\mathcal{Z}_K])) \otimes \Lambda_{\mathbf{k}}[n].$$

Then the proof just follows the line of the proof of Theorem 6.6. $\qquad \square$

Proof of Theorem 7.4. Let m be the vertex number of K, and m_i be the vertex number of K_i. First we do the case $\mathbf{k}$ is a field. From Theorem 4.2, we know that

$$\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})/([\mathcal{Z}_K]) \cong M_1 \times M_2 \times \cdots \times M_k \times \mathbf{k}^l,$$

where each $M_i \cong G^{n_i}(\widetilde{H}^*(\mathcal{Z}_{K_i}; \mathbf{k}))/([\mathcal{Z}_{K_i}] \otimes v_{[n_i]})$ with K_i flag, $n_i = m - m_i$ and each $\mathbf{k}$ summand in the above formula has trivial multiplication. Lemma 7.5 says that M_i are all graded indecomposable. Thus according to Theorem A.6 in the appendix, k, l are determined by $\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})$.

Clearly, K_i contains $(3m_i - 6)$ 1-faces, and so has $\binom{m_i}{2} - (3m_i - 6)$ missing faces. In other words,

$$\dim_{\mathbf{k}}(M_i^3) = \binom{m_i}{2} - (3m_i - 6) = \frac{m_i^2 - 7m_i}{2} + 6$$

(each missing face corresponds to a generator of $\widetilde{H}^3(\mathcal{Z}_{K_i}; \mathbf{k})$). Since $f(x) = \frac{x^2 - 7x}{2} + 6$ is a strictly monotone increasing function on $x \geq 4$, thus m_i $(1 \leq i \leq k)$ is determined by $\dim_{\mathbf{k}}(M_i^3)$. On the other hand, the vertex number m of K is determined by the top dimension d of $H^*(\mathcal{Z}_K; \mathbf{k})$: $m = d - 3$, so the cardinality k' of $\{j \in [n] \mid K_j = \partial \Delta^3\}$ equals $d - 3 - \sum_{i=1}^{k} m_i$. Therefore $n = k + k'$ is determined by $H^*(\mathcal{Z}_K; \mathbf{k})$, i.e., $n = n'$. Since

$\widetilde{H}^*(\mathcal{Z}_K; \mathbf{k})/([\mathcal{Z}_K]) \cong \widetilde{H}^*(\mathcal{Z}_{K'}; \mathbf{k})/([\mathcal{Z}_{K'}])$, then by combining Theorem 4.2, Corollary 6.14, Theorem A.6 and Lemma 7.5, we have up to a permutation

$$G^{n_i}(\widetilde{H}^*(\mathcal{Z}_{K_i}; \mathbf{k}))/([\mathcal{Z}_{K_i}] \otimes v_{[n_i]}) \cong G^{n_i}(\widetilde{H}^*(\mathcal{Z}_{K'_i}; \mathbf{k}))/([\mathcal{Z}_{K'_i}] \otimes v_{[n_i]}).$$

Since

$$G^{n_i}(\widetilde{H}^*(\mathcal{Z}_{K_i}; \mathbf{k}))/([\mathcal{Z}_{K_i}] \otimes v_{[n_i]}) \cong (\widetilde{H}^*(\mathcal{Z}_{K_i}; \mathbf{k})/([\mathcal{Z}_{K_i}])) \otimes \Lambda_{\mathbf{k}}[n_i],$$

it follows that $\widetilde{H}^*(\mathcal{Z}_{K_i}; \mathbf{k})/([\mathcal{Z}_{K_i}]) \cong \widetilde{H}^*(\mathcal{Z}_{K'_i}; \mathbf{k})/([\mathcal{Z}_{K'_i}])$.

For the integral case, the theorem can be proved in the same way, just by using Corollary A.7 instead of Theorem A.6.   □

Definition 7.6. A simplicial sphere K is called *strongly B-rigid* if whenever there is another simplicial sphere L such that $H^*(\mathcal{Z}_K)/([\mathcal{Z}_K]) \cong H^*(\mathcal{Z}_L)/([\mathcal{Z}_L])$, then $K \approx L$.

For flag spheres, we make the following

Conjecture 7.7. Every flag 2-sphere is strongly B-rigid, or more generally, every flag Gorenstein* complex is strongly B-rigid.

In [19], we have shown that this conjecture is true if K is a flag 2-sphere without 4-*belt* (an *n-belt* of K is a full subcomplex isomorphic to the boundary of an n-gon). Notice that if Conjecture 4.10 and Conjecture 7.7 are both true, then Theorem 7.4, together with Proposition 7.8 below, give a positive answer to Question A for moment-angle manifolds associated to simplicial 2-spheres.

Proposition 7.8. *Let $\mathcal{Z}_K$ be a moment-angle manifold associated to a simplicial 2-sphere K. If $H^*(\mathcal{Z}_K) \cong H^*(\mathcal{Z}_L)$ for a moment-angle manifold $\mathcal{Z}_L$, then L is a simplicial 2-sphere.*

Before proving Proposition 7.8, we introduce a fundamental fact from the polytope theory, known as the *Lower Bound Conjecture* (LBC), which was first proved by Barnette [4] for simplicial polytopes, and generalized to triangulated manifolds by Kalai [30].

Theorem 7.9 (LBC)**.** *Let K be a triangulated homology $(n-1)$-manifold with m vertices, and let e be the number of edges (1-simplices) of K. Then $e \geq mn - \binom{n+1}{2}$.*

proof of Proposition 7.8. The proof is by using LBC to show that if $\dim L > 2$, then

$$\text{rank } H^3(\mathcal{Z}_L) \neq \text{rank } H^3(\mathcal{Z}_K).$$

Let m and l be the vertex number of K and L respectively ($m \geq 4$ is obvious). From the analysis in the proof of Theorem 7.4, rank $H^3(\mathcal{Z}_K)$ equals the number of two-element missing faces of K, and so rank $H^3(\mathcal{Z}_K) = \binom{m}{2} - (3m - 6)$. If $\dim L = 1$, then $l = m + 1$, and rank $H^3(\mathcal{Z}_L) = \binom{m+1}{2} - (m+1)$. Thus

$$\text{rank } H^3(\mathcal{Z}_L) - \text{rank } H^3(\mathcal{Z}_K) = 3m - 7 > 0.$$

If $\dim L = n > 2$, since the top dimension d of $H^*(\mathcal{Z}_L)$ equals $n + l + 1 = m + 3$, then $n + 2 \leq l < m$. By LBC we have

$$\text{rank } H^3(\mathcal{Z}_L) \leq \binom{l}{2} - l(n+1) + \binom{n+2}{2} = \frac{l(l - 2n - 3)}{2} + \binom{n+2}{2}.$$

A straightforward calculation shows that the function $f(x) = \frac{x(x-2n-3)}{2}$ is strictly monotone increasing on $x \geq n + 2$. Thus

$$\text{rank } H^3(\mathcal{Z}_L) < \binom{m}{2} - m(n+1) + \binom{n+2}{2} = \binom{m}{2} - \frac{(n+1)(2m - 2 - n)}{2}.$$

Since the function $g(x) = \frac{(x+1)(2m-2-x)}{2}$ is strictly monotone increasing on $x \leq m - 2$, then

$$\text{rank } H^3(\mathcal{Z}_L) < \binom{m}{2} - (3m - 6) = \text{rank } H^3(\mathcal{Z}_K).$$

This is a contradiction, so it must have $n = 2$. $\qquad\square$

Now we go to another question: When a reducible 2-sphere K is B-rigid? Theorem 7.3 gives a necessary condition for K to be B-rigid.

Theorem 7.10. *Let K be a reducible simplicial 2-sphere. If K is B-rigid, then $K = T_4 \# T_4 \# T_4$ or $K = K_1 \# K_2$, where*

$$K_1 \in \{T_4, O_6, I_{12}\},$$
$$K_2 \in \{T_4, O_6, I_{12}, \xi_1(C_8), \xi_2(C_8), \xi_1(D_{20}), \xi_2(D_{20})\} \cup \{B_n : n \geq 7\}.$$

In fact this is also a sufficient condition for K to be B-rigid, this is shown in [19] by proving that the irreducible 2-spheres appear in Theorem 7.3 are all strongly B-rigid.

APPENDIX

A.1. **Poincaré duality and cup product pairing.** Let $\mathcal{R}$ be a commutative ring. For a closed $\mathcal{R}$-orientable n-manifold M ($[M]$ its fundamental class), consider the cup product pairing

$$H^k(M; \mathcal{R}) \times H^{n-k}(M; \mathcal{R}) \longrightarrow \mathcal{R}, \qquad (\phi, \psi) \mapsto (\phi \smile \psi)[M]$$

Such a bilinear pairing $A \times B \to \mathcal{R}$ is said to be *nonsingular* if the maps $A \to \text{Hom}_{\mathcal{R}}(B; \mathcal{R})$ and $B \to \text{Hom}_{\mathcal{R}}(A; \mathcal{R})$, obtained by viewing the pairing as a function of each variable separately, i.e., $a \mapsto \rho_a$, $\rho_a(b) = (a \smile b)[M]$, are both isomorphisms.

As we kown, the cup product pairing is nonsingular for closed $\mathcal{R}$-orientable manifolds when $\mathcal{R}$ is a field, or when $\mathcal{R} = \mathbb{Z}$ and torsion in $H^*(M; \mathbb{Z})$ is factored out. There is a generalization of this result to the case that when $\mathcal{R} = \mathbb{Z}_m$. First we need the following result which is a generalization of universal coefficient theorem for cohomology.

Theorem A.1. *Let $\mathcal{R}$ be a commutative ring. If $\mathcal{R}$ is an injective $\mathcal{R}$-module itself, then for any space X, there is an isomorphism*

$$h : H^k(X; \mathcal{R}) \cong \mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R}).$$

Proof. Let $(C_*(X), d)$ be the singular chain complex of X. Then

$$H^*(X; \mathcal{R}) = H^*(\mathrm{Hom}(C_*(X), \mathcal{R}), \delta) = H^*(\mathrm{Hom}_{\mathcal{R}}(C_*(X) \otimes \mathcal{R}, \mathcal{R}), \delta),$$

wherein δ is the dual coboundary map. From the proof universal coefficient theorem, we know that there is a natural homomorphism

$$h : H^*(\mathrm{Hom}_{\mathcal{R}}(C_*(X) \otimes \mathcal{R}, \mathcal{R}), \delta) \to \mathrm{Hom}_{\mathcal{R}}(H_*(C_*(X) \otimes \mathcal{R}, d), \mathcal{R}).$$

The elements of $\mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R})$ can be represented by $\mathcal{R}$-homomorphisms $\varphi :$ $\mathrm{Ker}\, d_k \to \mathcal{R}$ that vanish on $\mathrm{Im}\, d_{k+1}$. Since $\mathcal{R}$ is injective, φ can be extended to an $\mathcal{R}$-homomorphism $\bar\varphi : C_k \otimes \mathcal{R} \to \mathcal{R}$ that still vanish on $\mathrm{Im}\, d_{k+1}$. Thus we have an $\mathcal{R}$-homomorphism $\mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R}) \to \mathrm{Ker}\, \delta_k$. Composing this with the quotient map $\mathrm{Ker}\, \delta_k \to H^k(X; \mathcal{R})$ gives a homomorphism from $\mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R})$ to $H^k(X; \mathcal{R})$. If we follow this map by h we get the identity map on $\mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R})$. Hence by the same reasoning as in the proof of universal coefficient theorem, we have the split exact sequence

$$0 \to \mathrm{Ext}_{\mathcal{R}}(H_{k-1}(X; \mathcal{R}), \mathcal{R}) \to H^k(X; \mathcal{R}) \xrightarrow{h} \mathrm{Hom}_{\mathcal{R}}(H_k(X; \mathcal{R}), \mathcal{R}) \to 0.$$

Note that $\mathrm{Ext}_{\mathcal{R}}(H_{k-1}(X; \mathcal{R}), \mathcal{R}) \equiv 0$ since $\mathcal{R}$ is injective. We get the desired result. □

According to Baer Criterion, $\mathbb{Z}_m$ is an injective $\mathbb{Z}_m$-module, so we immediately get the following

Corollary A.2. *For any integer $m \geq 2$, there is an isomorphism*

$$H^k(X; \mathbb{Z}_m) \cong \mathrm{Hom}_{\mathbb{Z}_m}(H_k(X; \mathbb{Z}_m), \mathbb{Z}_m).$$

Let $\eta : H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{Z}_m)$ (resp. $\eta' : H_*(M; \mathbb{Z}) \to H_*(M; \mathbb{Z}_m)$) be the homomorphism induced by the map $\mathbb{Z} \to \mathbb{Z}_m$ reducing coefficients mod m. It is easy to check that $\mathrm{Im}\, \eta \cong H^*(M; \mathbb{Z}) \otimes \mathbb{Z}_m$ (resp. $\mathrm{Im}\, \eta' \cong H_*(M; \mathbb{Z}) \otimes \mathbb{Z}_m$) is a direct summand of $H^*(M; \mathbb{Z}_m)$ (resp. $H_*(M; \mathbb{Z}_m)$). The proof of the following theorem is just by following the way in [26, Proposition 3.38].

Proposition A.3. *(i) If $\mathcal{R}$ is an injective $\mathcal{R}$-module itself, then the cup product pairing is nonsingular for closed $\mathcal{R}$-orientable n-manifolds.*

(ii) For closed orientable manifolds, the cup product pairing (with $\mathbb{Z}_m$ coefficient) restricted to $\mathrm{Im}\, \eta \times (H^*(M; \mathbb{Z}_m)/\mathrm{Ker}\, \widetilde{h})$ and $(H^*(M; \mathbb{Z}_m)/\mathrm{Im}\, \eta) \times \mathrm{Ker}\, \widetilde{h}$ are both nonsingular, where $\widetilde{h}$ is the map in the split exact sequence of universal coefficient theorem:

$$0 \to \mathrm{Ext}(H_{k-1}(M; \mathbb{Z}), \mathbb{Z}_m) \to H^k(M; \mathbb{Z}_m) \xrightarrow{\widetilde{h}} \mathrm{Hom}(H_k(M; \mathbb{Z}), \mathbb{Z}_m) \to 0.$$

Proof. For (i), consider the composition

$$H^{n-k}(M;\mathcal{R}) \xrightarrow{h} \mathrm{Hom}_{\mathcal{R}}(H_{n-k}(M;\mathcal{R}),\mathcal{R}) \xrightarrow{D^*} \mathrm{Hom}_{\mathcal{R}}(H^k(M;\mathcal{R});\mathcal{R}),$$

where h is the map in Theorem A.1 and D^* is the Hom dual of the Poincaré duality map $D: H^k \to H_{n-k}$, $\phi \mapsto [M] \frown \phi$. The composition D^*h sends $\psi \in H^{n-k}(M;R)$ to the homomorphism $\phi \mapsto \psi([M] \frown \phi) = \phi \smile \psi([M])$. The first statement follows by Theorem A.1 immediately.

For (ii), consider the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}(H_{k-1}(M;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{Z}) & \xrightarrow{h} & \mathrm{Hom}(H_k(M;\mathbb{Z}),\mathbb{Z}) \\
\eta \downarrow & & \eta \downarrow & & \eta \downarrow \\
\mathrm{Ext}(H_{k-1}(M;\mathbb{Z}),\mathbb{Z}_m) & \longrightarrow & H^k(M;\mathbb{Z}_m) & \xrightarrow{\widetilde{h}} & \mathrm{Hom}(H_k(M;\mathbb{Z}),\mathbb{Z}_m)
\end{array}$$

It is easy to see that η restricted to $\mathrm{Ker}\, h = \mathrm{Ext}(H_{k-1}(M;\mathbb{Z}),\mathbb{Z})$ is surjective. Hence for any $\phi \in \mathrm{Im}\,\eta$, $\psi \in \mathrm{Ker}\,\widetilde{h}$, there exist $\phi' \in H^{n-k}(M;\mathbb{Z})$ and $\psi' \in \mathrm{Ker}\,h$ such that $\phi = \eta(\phi')$ and $\psi = \eta(\psi')$. On the other hand, for such ϕ', ψ', $[M] \frown \phi' \in H_k(M;\mathbb{Z})$, so $\phi' \smile \psi'([M]) = \psi'([M] \frown \phi') = 0$, i.e. $\phi' \smile \psi' = 0$. Thus $\phi \smile \psi = \eta(\phi' \smile \psi') = 0$. So the cup product pairings in (ii) are well defined.

Now we consider the following commutative diagram

$$\begin{array}{ccc}
H^k(M;\mathbb{Z}) & \xrightarrow{D:=[M]\frown} & H_{n-k}(M;\mathbb{Z}) \\
\eta \downarrow & & \eta' \downarrow \\
H^k(M;\mathbb{Z}_m) & \xrightarrow{D:=[M]\frown} & H_{n-k}(M;\mathbb{Z}_m)
\end{array}$$

Since M is orientable, the maps in the rows are isomorphisms. It is easy to check that the map in the bottom row restricted to $\mathrm{Im}\,\eta \xrightarrow{[M]\frown} \mathrm{Im}\,\eta'$ is an isomorphism. Therefore since the following diagram is commutative,

$$\begin{array}{ccc}
H^k(M;\mathbb{Z}_m) & \xrightarrow[\cong]{h} & \mathrm{Hom}(H_k(M;\mathbb{Z}_m),\mathbb{Z}_m) \\
 & \searrow^{\widetilde{h}} & \downarrow^{(\eta')^*} \\
 & & \mathrm{Hom}(H_k(M;\mathbb{Z}),\mathbb{Z}_m)
\end{array}$$

we have

$$H^k(M;\mathbb{Z}_m)/\mathrm{Ker}\,\widetilde{h} = \mathrm{Hom}(H_k(M;\mathbb{Z}_m),\mathbb{Z}_m)/\mathrm{Ker}\,(\eta')^* = \mathrm{Hom}(\mathrm{Im}\,\eta',\mathbb{Z}_m).$$

Composing this with the Hom dual of the Poincaré duality map $D^*: \mathrm{Hom}(\mathrm{Im}\,\eta',\mathbb{Z}_m) \to \mathrm{Hom}(\mathrm{Im}\,\eta,\mathbb{Z}_m)$, we get the desired isomorphism $H^k(M;\mathbb{Z}_m)/\mathrm{Ker}\,\widetilde{h} \to \mathrm{Hom}(\mathrm{Im}\,\eta,\mathbb{Z}_m)$. Meanwhile, the composition

$$\mathrm{Im}\,\eta \xrightarrow{D} \mathrm{Im}\,\eta' \xrightarrow{\lambda} \mathrm{Hom}(\mathrm{Hom}(\mathrm{Im}\,\eta',\mathbb{Z}_m),\mathbb{Z}_m) = \mathrm{Hom}(H^k(M;\mathbb{Z}_m)/\mathrm{Ker}\,\widetilde{h},\,\mathbb{Z}_m)$$

gives another isomorphism, where λ is the double dual of $\mathrm{Im}\,\eta$, i.e. $\lambda(x)(\phi) = \phi(x)$ for $x \in \mathrm{Im}\,\eta'$, $\phi \in \mathrm{Hom}(\mathrm{Im}\,\eta', \mathbb{Z}_m)$.

The nonsingularity for $(H^*(M; \mathbb{Z}_m)/\mathrm{Im}\,\eta) \times \mathrm{Ker}\,\widetilde{h}$ is similar. $\qquad\square$

Let M be an orientable n-manifold, $\mathcal{A}$ be a graded ideal of $H^*(M)$ satisfies that $\mathcal{A}$ is a direct summand of $H^*(M)$ and the cup product pairing restricted to $\mathcal{A}/Torsion(\mathcal{A})$ is nonsingular. So there exists a graded subgroup $\mathcal{B} \subset H^*(M)$ such that $H^*(M) = \mathcal{A} \oplus \mathcal{B}$ (as groups). However, the choice of $\mathcal{B}$ satisfying the above condition is not unique, and the cup product of $\mathcal{A}$ and $\mathcal{B}$ is not zero in general.

Proposition A.4. *Let M, $\mathcal{A}$, $\mathcal{B}$ be as above, $\mathcal{T} = Torsion(H^*(M; \mathbb{Z}))$. Suppose m is the exponent of $\mathcal{T}$. If there exists a subring Γ of $H^*(M; \mathbb{Z}_m)$ such that $\eta(\mathcal{A}) \subset \Gamma$ and $H^*(M; \mathbb{Z}_m) = \Gamma \oplus \eta(\mathcal{B})$ for some $\mathcal{B}$, then $\mathcal{B}$ can be chosen properly such that $\mathcal{A} \smile \mathcal{B} = 0$.*

Proof. Firstly, let us arbitrarily take $\mathcal{B} \subset H^*(M)$ so that $H^*(M) = \mathcal{A} \oplus \mathcal{B}$. Set $\widetilde{\mathcal{A}} = \mathcal{A}/Torsion(\mathcal{A})$, and chose a homogenous basis $\{b_i\}_{i=1}^s$ for $\mathcal{B}$, $b_i \in \mathcal{B}^{k_i}$. As defined at the beginning of the appendix, $\rho_{b_i} \in \mathrm{Hom}(\widetilde{\mathcal{A}}^{n-k_i}, \mathbb{Z})$. Since the cup product pairing on $\widetilde{\mathcal{A}}$ is nonsingular, then there exists an $a_i \in \mathcal{A}^{k_i}$ such that $\rho_{a_i} = \rho_{b_i}$. Let $b_i' = b_i - a_i$, then $b_i' \smile \mathcal{A}^{n-k_i} = 0$. We claim that $b_i' \smile \mathcal{A} \subset Torsion(\mathcal{A})$. Otherwise $b_i' \smile a \neq 0 \in \widetilde{\mathcal{A}}^{k_i+j}$ for some $a \in \mathcal{A}^j$, $j < n - k_i$. Still from the nonsingularity of $\widetilde{\mathcal{A}}$, there is an $a' \in \mathcal{A}^{n-k_i-j}$ such that $b_i' \smile a \smile a' \neq 0$, this is a contradiction. Let $\mathcal{B}' = \langle b_i' \rangle_{i=1}^s$ be the free abelian group generated by b_i', then apparently $H^*(M) = \mathcal{A} \oplus \mathcal{B}'$.

Now we consider the cohomology ring with coefficient $\mathbb{Z}_m$. It is easy to see that $\eta(\mathcal{A})$ is a direct summand of Γ. Suppose $\Gamma = \mathcal{C} \oplus \eta(\mathcal{A})$, then $\mathcal{C} = H^*(M; \mathbb{Z}_m)/\mathrm{Im}\,\eta$. Thus the restriction map $\rho_{\eta(b_i')} \mid \mathcal{C}$ is an element of $\mathrm{Hom}_{\mathbb{Z}_m}(H^*(M; \mathbb{Z}_m)/\mathrm{Im}\,\eta, \mathbb{Z}_m)$. According to Proposition A.3 (ii), there is an element $\gamma_i \in \mathrm{Ker}\,\widetilde{h}$ such that $\rho_{\gamma_i} \mid \mathcal{C} = \rho_{\eta(b_i')} \mid \mathcal{C}$. From the proof of Proposition A.3 (ii) we know that $\mathrm{Ker}\,\widetilde{h} = \eta(\mathrm{Ker}\,h)$. $\mathrm{Ker}\,h = \mathcal{T}$ by the assumption that $\mathcal{T}$ has exponent m. Hence there exists an $a_i' \in \mathcal{T}$ such that $\eta(a_i') = \gamma_i$. Now let $b_i'' = b_i' - a_i'$. Clearly $b_i'' \smile \mathcal{A}^{n-k_i} = 0$, so $\rho_{\eta(b_i'')} \mid \eta(\mathcal{A}) = 0$. Meanwhile, $\rho_{\eta(b_i'')} \mid \mathcal{C} = \rho_{\eta(b_i')} - \rho_{\gamma_i} \mid \mathcal{C} = 0$. So $\rho_{\eta(b_i'')} \mid \Gamma = 0$. We claim that $b_i'' \smile \mathcal{A} = 0$ for each i. Otherwise $b_i'' \smile a \neq 0 \in Torsion(\mathcal{A}^{k_i+j}) \subset \mathrm{Ker}\,h$ for some $a \in \mathcal{A}^j$, $j < n - k_i$. It follows that $\eta(b_i'' \smile a) \neq 0 \in \mathrm{Ker}\,\widetilde{h}$. By applying Proposition A.3 (ii) again, we can find $c \in \mathcal{C}^{n-k_i-j}$ such that $\eta(b_i'') \smile \eta(a) \smile c \neq 0 \in H^n(M; \mathbb{Z}_m)$. However, since Γ is a subring, $c \smile \eta(a) \in \Gamma$, contradict to the fact $\rho_{\eta(b_i'')} \mid \Gamma = 0$. Finally, let $\mathcal{B}'' = \langle b_i'' \rangle_{i=1}^s$, then we have $H^*(M) = \mathcal{A} \oplus \mathcal{B}''$ and $\mathcal{A} \smile \mathcal{B}'' = 0$. $\qquad\square$

A.2. **Algebra version of Krull-Schmidt Theorem.** This appendix is devoted to generalizing a famous theorem in group and module theory, which is called "Krull-Schmidt theorem" (for a while the theorem was also called the "Wedderburn-Remak-Krull-Schmidt theorem") to the ring or algebra case. This theorem can be stated as below (cf. [29] p.115). A module M is *decomposable* if $M = M_1 \oplus M_2$ where $M_i \neq 0$. Otherwise, M is called *indecomposable*.

Krull-Schmidt Theorem. *Let M be a module that is both artinian and noetherian and let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n = N_1 \oplus N_2 \oplus \cdots \oplus N_m$ where the M_i and N_j are indecomposable. Then $m = n$ and there is a permutation $i \curvearrowright i'$ such that $M_i \cong N_{i'}$, $1 \leq i \leq n$.*

A natural and similar question is that: Suppose a ring $\mathcal{R}$ is a direct product of a finite number of indecomposable rings. Are the components unique? Now by following the line of the proof of this theorem and by adding some supplementary arguments, we can show that there is a similar condition to assure this.

If f is an endomorphism of a ring $\mathcal{R}$, we define

$$f^\infty \mathcal{R} = \bigcap_{n=0}^\infty f^n(\mathcal{R}) \quad \text{and} \quad f^{-\infty} 0 = \bigcup_{n=0}^\infty \operatorname{Ker} f^n.$$

The following result is the ring version of Fitting's Lemma (cf. [29] p.113).

Lemma A.5. *Let f be an endomorphism of a ring $\mathcal{R}$ that is both artinian and noetherian. If $f^n(\mathcal{R})$ is an ideal of $\mathcal{R}$ for every $n \geq 0$, then we have the Fitting decomposition*

$$\mathcal{R} = f^\infty \mathcal{R} \times f^{-\infty} 0.$$

Proof. Since $\mathcal{R}$ is artinian, there is an integer s such that $f^s(\mathcal{R}) = f^{s+1}(\mathcal{R}) = \cdots = f^\infty \mathcal{R}$. Since $\mathcal{R}$ is noetherian, there exists t such that $\operatorname{Ker} f^t = \operatorname{Ker} f^{t+1} = \cdots = f^{-\infty} 0$. Let $r = \max\{s, t\}$, so $f^\infty \mathcal{R} = f^r(\mathcal{R})$ and $f^{-\infty} 0 = \operatorname{Ker} f^r$. If $z \in f^\infty \mathcal{R} \cap f^{-\infty} 0$, so $z = f^r y$ for some $y \in \mathcal{R}$. Then $0 = f^r z = f^{2r} y$ and $y \in \operatorname{Ker} f^{2r} = \operatorname{Ker} f^r$. Hence $z = f^r y = 0$. Thus $f^\infty \mathcal{R} \cap f^{-\infty} 0$.

Now let $x \in \mathcal{R}$. Then $f^r x \in f^r(\mathcal{R}) = f^{2r}(\mathcal{R})$, so $f^r x = f^{2r} y$ for some $y \in \mathcal{R}$. Then $f^r(x - f^r y) = 0$ and so $z = x - f^r y \in f^{-\infty} 0$. Then $x = f^r y + z$ and $f^r y \in f^\infty \mathcal{R}$. Hence there is an additively splitting $\mathcal{R} = f^\infty \mathcal{R} \oplus f^{-\infty} 0$. From the hypothesis that $f^r(\mathcal{R})$ is an ideal of $\mathcal{R}$, we get the desired decomposition. $\square$

Now we give the algebra version of Krull-Schmidt Theorem as follows.

Theorem A.6. *Let R, Q be two finitely dimensional $\mathbf{k}$-algebras with $\mathbf{k}$ a field and let*

$$R = R_1 \times R_2 \times \cdots \times R_n,$$
$$Q = Q_1 \times Q_2 \times \cdots \times Q_m,$$

where the R_i and Q_j are indecomposable $\mathbf{k}$-algebras. If $R \cong Q$, then $m = n$ and there is a permutation $i \curvearrowright i'$ such that $R_i \cong Q_{i'}$, $1 \leq i \leq n$. Moreover, if R, Q are isomorphic graded $\mathbf{k}$-algebras and the R_i and Q_j are graded indecomposable, then the isomorphisms $R_i \cong Q_{i'}$ are graded isomorphisms.

Proof. Let $i_s : R_s \to R$ and $j_t : Q_t \to Q$ be the respective inclusions and let $p_s : R \to R_s$, $q_t : Q \to Q_t$ be the respective projections. Suppose $\phi : R \to Q$ is an isomorphism. Define

$e_t = q_t \phi i_1 : R_1 \to Q_t$ and $f_t = p_1 \phi^{-1} j_t : Q_t \to R_1$. Then $g_t = f_t e_t$ is an endomorphism of R_1. Note that

$$\sum_{t=1}^{m} g_t = p_1 \phi^{-1} (\sum_{t=1}^{m} j_t q_t) \phi i_1 = p_1 \phi^{-1} 1_R \phi i_1 = 1_{R_1}.$$

So we have

$$R_1 = g_1(R_1) + g_2(R_1) + \cdots + g_m(R_1) \text{ (as } \mathbf{k}\text{-module)}.$$

On the other hand, since $Q_i \cdot Q_j = 0$ $(i \neq j)$, then for $x_i = g_i(y_i) \in g_i(R_1)$, $x_j = g_j(y_j) \in g_j(R_1)$ we have that

$$x_i x_j = p_1 \phi^{-1}(y_i) \cdot p_1 \phi^{-1}(y_j) = p_1 \phi^{-1}(y_i y_j) = 0. \tag{A.1}$$

This implies that $g_t(R_1)$ is an ideal of R_1 for all t. Similarly, we have that $g_t{}^k(R_1)$ is an ideal of R_1 for all $k > 0$.

We claim that one of g_t is an automorphism of R_1. We prove this by induction on m. If $m = 1$, there is nothing to prove. For the induction step, we may assume g_1 is nilpotent (otherwise g_1 is an automorphism by Lemma A.5 and the indecomposability of R_1). It is easy to see by equation (A.1) that $h = 1 - g_1 = g_2 + \cdots + g_m$ is an endomorphism of R_1. Since $g_1{}^r = 0$ for some r, then

$$(1 - g_1)(1 + g_1 + \cdots + g_1{}^{r-1}) = 1 = (1 + g_1 + \cdots + g_1{}^{r-1})(1 - g_1).$$

Thus h is an automorphism of R_1. Let $g_i' = g_i h^{-1}$, then $g_2' + \cdots + g_m' = 1$. Hence by induction there is a t $(2 \leq t \leq m)$ such that g_t' is an automorphism of R_1, and so is g_t.

Assume g_t is an automorphism of R_1, now we prove that $e_t : R_1 \to Q_t$ is an isomorphism. It is clear that e_t is a monomorphism and f_t restricted to $e_t(R_1)$ is an isomorphism. It follows that Q_t additively splits as $e_t(R_1) \oplus \mathrm{Ker}\, f_t$. Moreover, by a similar analysis as before, we have $e_t(R_1)$ is an ideal of Q_t. Thus there is a multiplicative splitting $Q_t = e_t(R_1) \times \mathrm{Ker}\, f_t$. The fact that Q_t is indecomposable implies that $Q_t = e_t(R_1)$, i.e. e_t is an isomorphism.

Define $e_{s,t} = q_t \phi i_s : R_s \to Q_t$ (thus $e_{1,t} = e_t$ in terms of the former notation). Let

$$\mathcal{N}_i = \{t \in [m] \mid e_{i,t} \text{ is an isomorphism}\}.$$

By the preceding argument, $\mathcal{N}_i \neq \varnothing$ for each $i \in [n]$. Next we show that if R_i has nontrivial multiplication then $\mathcal{N}_i \cap \mathcal{N}_j = \varnothing$ for $j \neq i$. To see this, suppose $R_i \cong Q_t$ has nontrivial multiplication, then there are two elements $a, b \in Q_t$ such that $ab \neq 0$. Suppose on the contrary that there exists $j \neq i$ such that $e_{j,t}$ is an isomorphism, then there are $x \in R_i$, $y \in R_j$ so that $q_t \phi(x) = a$, $q_t \phi(y) = b$. However $ab = q_t \phi(xy) = 0$ since $xy = 0$, a contradiction.

Suppose n' is the number of R_i which has nontrivial multiplication, and m' is the number of Q_j which has nontrivial multiplication. Hence $\mathcal{N}_i \cap \mathcal{N}_j = \varnothing$ $(j \neq i)$ implies that $n' \leq m'$. From the symmetry of R and Q, we also have $m' \leq n'$, then $m' = n'$. Thus up to a permutation of n' there is an 1-to-1 correspondence between the indecomposable factors with nontrivial multiplication of R and Q: $R_i \cong Q_i$. On the other hand, since $\mathbf{k}$

is a field, any indecomposable **k**-algebra with trivial multiplication is an 1-dimensional **k**-vector space. Thus the theorem follows by the fact that the dimensions of R and Q over **k** are equal.

The graded case can be proved in the same way. $\qquad\square$

Corollary A.7. *Let $R \cong Q$ be torsion-free rings with finite ranks and let*

$$R = R_1 \times R_2 \times \cdots \times R_n,$$
$$Q = Q_1 \times Q_2 \times \cdots \times Q_m,$$

where the R_i and Q_j are indecomposable rings. If the $R_i \otimes \mathbf{k}$ and $Q_j \otimes \mathbf{k}$ are indecomposable $\mathbf{k}$-algebras whenever $\mathbf{k} = \mathbb{Q}$ or $\mathbb{Z}_p$ (p is a prime), then $m = n$ and there is a permutation $i \curvearrowright i'$ such that $R_i \cong Q_{i'}$, $1 \le i \le n$.

Proof. Since $R \cong Q$, then $R \otimes \mathbb{Q} \cong Q \otimes \mathbb{Q}$. According to Theorem A.6, $m = n$ and $R_i \otimes \mathbb{Q} \cong Q_i \otimes \mathbb{Q}$ up to a permutation of $[n]$. Let $e_{i,j} : R_i \to Q_j$ be as in the proof of Theorem A.6. Then by the arguments in the proof of Theorem A.6, if R_i has nontrivial multiplication, then $e_{i,i} \otimes 1_{\mathbb{Q}}$ is an isomorphism and $e_{i,j} \otimes 1_{\mathbb{Q}}$ is not an isomorphism for $i \ne j$. Thus if R_i has nontrivial multiplication, then

  (1) $\operatorname{Ker} e_{i,i} = 0$ and $\operatorname{rank} e_{i,i}(R_i) = \operatorname{rank} Q_i$, which implies that $\operatorname{coker} e_{i,i}$ is finite;
  (2) $\operatorname{Ker} e_{i,j} \ne 0$ or $\operatorname{rank} e_{i,j}(R_i) < \operatorname{rank} Q_j$ for $i \ne j$, which implies that $e_{i,j} \otimes 1_{\mathbb{Z}_p}$ is not an isomorphism for $i \ne j$ and any prime p.

On the other hand, from the proof of Theorem A.6 we know that there must be a $j \in [m]$ such that $e_{i,j} \otimes 1_{\mathbb{Z}_p}$ is an isomorphism. So $e_{i,i} \otimes 1_{\mathbb{Z}_p}$ is actually an isomorphism. However if $\operatorname{coker} e_{i,i} \ne 0$, say $\mathbb{Z}_p$, then $\operatorname{Ker} e_{i,i} \otimes 1_{\mathbb{Z}_p} \ne 0$, a contradiction. Hence $\operatorname{coker} e_{i,i} = 0$ and $e_{i,i}$ is an isomorphism.

Therefore there is an 1-to-1 correspondence between the indecomposable factors with nontrivial multiplication of R and Q: $R_i \cong Q_i$. On the other hand, note that any torsion-free indecomposable ring with trivial multiplication is additively isomorphic to $\mathbb{Z}$. Thus the corollary follows. $\qquad\square$

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Feifei Fan, School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China

 E-mail address: `fanfeifei@mail.nankai.edu.cn`

Xiangjun Wang, School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, P. R. China

 E-mail address: `xjwang@nankai.edu.cn`